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THREE-HINGED MASONRY ARCHES; LONG SPANS ESPECIALLY CONSIDERED.

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WITH DISCUSSION.

Introductory.

The advantages which well-designed masonry arches offer, as compared with the less durable structures of iron and steel, have been adequately demonstrated by modern experience.

The cost of maintenance of iron and steel bridges, together with their more or less limited lasting qualities, are sometimes offset by the ease, simplicity and accuracy of design and erection to which they are susceptible. The time allowable for construction may also, in many cases, weigh strongly in their favor.

However, the many masonry arches built centuries ago—a few antedating written history—are indisputable evidences of permanency. Few of these arches have required any repairs, and their cost of maintenance has amounted to almost nothing, a fact not to be realized in metal bridges.

The purpose of this paper is to demonstrate that masonry arches may be constructed on any good foundation, such as hard clay, and that they will admit of an accuracy and simplicity of design quite equal to that attainable for similar structures of iron or steel. In many instances concrete arches are even cheaper in first cost than metal bridges. Besides, the former possess the additional advantages of permanency and low cost of maintenance.

Recent progress, achieved through the earnest labors of German, French and Austrian engineers, has destined the masonry arch to become the successful competitor of iron and steel bridges, whenever the natural conditions of foundations and length of span do not offer unsurmountable difficulties.

The great advances accomplished in the manufacture of cements during the past ten years, the elaborate arch tests made by the Austrian Society of Engineers and Architects from 1890 to 1895, and the construction of a few three-hinged masonry and concrete arches, venturing the adoption of high unit stresses, low factors of safety and long spans; these mark the arrival of a new era in masonry bridge construction.

However, many difficulties are encountered in the construction of fixed masonry arches, owing particularly to insufficient elasticity in the masonry. The natural deformations in the arch, caused by shrinkage of the masonry, setting of mortar, stress and temperature, usually cause cracks which, while rarely of a serious character, are reasons for discouragement to the engineer, who has probably applied every known precaution to prevent their occurrence.

According to the recommendations of the Austrian Society of Engineers and Architects, as a result of their elaborate tests, a fixed masonry arch should be constructed only when the following conditions can be realized:

1. The abutments must be absolutely rigid.

2. The falsework must retain its form during the construction of the arch ring.

3. The material (stone and mortar) must be of the best quality.

4. The construction of the arch ring must be most carefully conducted.

5. The falsework must not be released until the mortar has thoroughly set.

6. When the falsework is released it must be done gradually and uniformly.

These conditions, except the two first named, can always be fulfilled, though the lack of rigidity of abutments and falsework are the two great obstacles in the way of long-span masonry arches without hinges.

In matters pertaining to the design of fixed masonry arches, it is safe to say that the method based on the theory of elasticity is the only one entitled to full confidence, and permitting of an analysis corresponding in accuracy with the knowable properties of the material. All other methods are too approximate to admit of close designing, such as the modern status of engineering science would generally demand.

This modern and most exact method, however, is not free from criticism. While the fundamental principles of the theory are almost axiomatic, their final application to the solution of stresses is extremely complicated, so much so that few engineers can be credited with the patience and earnest endurance to master either the method or the solution of a problem to which it is applied.

Therefore, unless the masonry arch can be so treated as to combine clearness, simplicity, undoubted accuracy and economy in design with faultless construction, the field of usefulness of this class of structure will remain restricted, and such monuments as the Cabin John Bridge will continue to remain curiosities of rare production.

This is not what the masonry arch deserves, in view of its practically everlasting life, nominal cost of maintenance and naturally æsthetic form, which latter should be a prime factor, though rarely given much consideration, in the choice of a bridge.

Essentially all the harassing features of fixed masonry arches are overcome by the introduction of hinges at the crown and abutments, thus permitting a rigid analytical treatment and affording almost absolute safety against cracks, even though small settlements may take place in the abutments. The idea was introduced by Koepke, of Dresden, in 1880, by providing open joints at crown and haunches. Karl v. Leibbrand, Stuttgart, in 1885, substituted sheet lead for the open joints, and in 1893, applied cast-iron, hinged bearings. The author, as early as 1888, while engaged on the construction of the strategical railway Weizen-Immendingen, Baden, Germany, advocated metal hinges for masonry arches, but prejudice and custom prevented a practical application being made at that time.

Some of the noteworthy bridges which have been constructed with hinges or hinge-like joints are briefly described in the following:

1. Bridges on the railroads of Saxony, built in 1880. by Koepke. The largest was of sandstone, 13 m. span, 3 m. rise, thickness of arch ring 0.50 m. to 0.60 m. Hinges consisted of a convex surface of sandstone, radius = 0.977 m., rolling in a concave surface, radius = 1.105 m. The maximum unit stress was 12.87 atm. Several arches of this type were constructed, some with only two hinges and some of concrete. All gave excellent satisfaction.

2. Sandstone bridge over the Enz River near Hoefen, Germany, built by Leibbrand, in 1885. Span, 28 m.; rise, 2.8 m.; maximum stress, 24 atm. Hinge-like joints of sheet lead. Several other bridges of this type were built in 1886 to 1890. The unit stresses were successively increased until 56.4 atm. were attained on the Forbach Bridge, in Baiersbronn, using sandstone of 653 atm. breaking strength.

3. Concrete arch over the Danube River, near Munderkingen, Wurtemberg, built by K. v. Leibbrand, in 1893. Span, 50 m.; rise, 5 m.; thickness of arch, 1 m. at crown, 1.4 m. at quarter points, and 1.1 m. at abutments. This arch was constructed as a three-hinged arch, and was the first masonry arch with actual hinged joints. The maximum compression in the arch was 34.6 atm. and 57 atm. adjacent to the steel hinges. The concrete was composed of 1 part Portland cement to 2½ parts sand and 5 parts broken limestone, showing an ultimate compressive strength of 254 atm. in 28 days and 520 atm. in 2 years and 7 months. The settlement at the crown, from the time of closing the arch ring to the entire completion, was 13.1 cm. One abutment is founded on rock, the other has a pile foundation.

4. Concrete bridge over the Danube near Rechtenstein, Wurtemberg, built in 1893, by Engineer Braun. This bridge is made up of two arches, each of 23 m. span and 2.5 m. rise; the thickness of the arch is 0.65 m. at the crown and 0.9 m. at the haunches. Concrete for arch ring was composed of 1 part Portland cement to 2½ parts sand to 5 parts gravel and ½ part quarry stone. The hinges are 18-cm. and 20-cm. lead strips for the crown and abutments, respectively. The highest stress in the arch ring was 18 atm., and the settlements at the crowns of the two spans were 4.0 and 3.0 cm. One abutment was founded on piles, the other on gravel and boulder strata, and the middle pier on solid rock.

5. Bridge de la Coulouvrenière, over the Rhône River, in Geneva, Switzerland, built in 1895, by Engineer Butticaz. This bridge was made of concrete and consists of two main arches spanning 40 m. each, with a rise of 5.55 m. each, separated by a small span of 14 m., and a 12-m. arch adjoining the abutment of one of the main arches. The large spans were patterned after the arch at Munderkingen, and the small spans were supplied with lead joints, as the bridge at Rechtenstein.

6. Concrete bridge over the Danube, near Inzighofen, Wurtemberg, built in 1896, by Max Leibbrand. Span, 43 m.; rise, 4.46 m. hinged

at crown and abutments with cast-iron, hinged pedestals. Thickness of arch ring, 0.7 m. at crown, 1.1 m. at quarter points and 0.78 m. at abutments. Maximum stress, 36.5 atm. in compression, and 1 atm. in tension. Concrete for arch ring was composed of 1 part Portland cement to $2\frac{1}{2}$ parts sand to 4 parts crushed limestone with $\frac{1}{2}$ part limestone screenings. Settlement of arch during construction (4 months) was 8 cm.

All the above-mentioned hinged and semi-hinged arches, besides others which could not be enumerated here, have given excellent satisfaction and have developed no cracks, even though some were founded on piles and others on clay foundations. Age will undoubtedly be beneficial rather than detrimental, which has never been said for iron or steel bridges.

The oldest concrete bridge seems to have been built in Switzerland near Aarau, in 1840, using Roman cement. This bridge has a span of 7.2 m., and a rise of 3 m. Even this cement, which is not as good as most natural cements, has stood the test of time.

With the adoption of three hinges and the evidence just submitted, it will be possible to construct a masonry arch on almost any moderately good foundation and with reasonable assurance against cracks, both during and after construction, all of which should be regarded as a welcome step in advance. This feature also makes it possible to determine the stresses, for any system of loading, with accuracy and certainty, also to stress the material from one-tenth to one-sixth its ultimate strength as obtained from test samples. All these advantages combined in the three-hinged masonry arch place it on a high plane of engineering perfection. It is hoped that this paper may be the means of introducing this form of arch construction into the United States.

In the following, all dimensions are given in the metric system simply for convenience in computations, but the formulas are equally applicable to any other system of units. The abbreviations used are:

1 meter	= 1 m.	= 3.2809 ft.
1 square meter	= 1 m.^{2}	= 10.7641 sq. ft.
1 cubic meter	= 1 m.^{3}	= 35.3156 cu. ft.
1 centimeter	= 1 cm.	= 0.3937 in.
1 kilogram	= 1 kgr.	= 2.2046 lbs. avoir.
1 atmosphere	= 1 atm.	= 0.9482 English atm. = 1 kgr. per cm.^{2} = 14.223 lbs. per square inch.

Theoretical Deductions.

I.—General Equations for the Three-Hinged Arch.

(*a*) *Applied Forces and Reactions.*—Given the arch, Fig. 1, with hinged joints at A, B and O, acted upon by the forces P_1 and P_2, assumed to represent the resultants of all vertical loads applied respectively to the left and right of the crown O. Required to find the reactions at the hinges A, B and O.

The thrust produced by P_1 on the segment $O\,B$ must pass through the hinge O, the only point of contact between the segments $A\ O$ and $O\ B$. This thrust must also pass through the hinge B, otherwise the segment $O\ B$ would rotate about B. Also, the intersection c_1 of $O\ B$ with the force P_1 must be a point on the line of the reaction produced

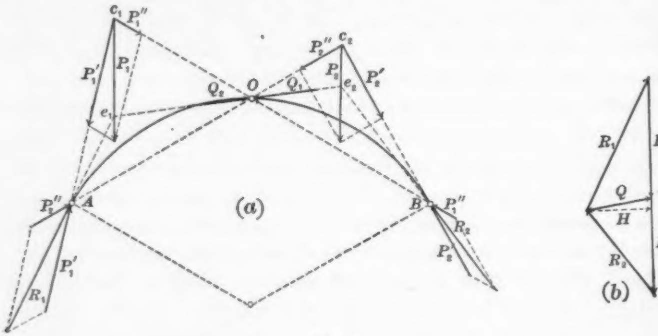

Fig. 1.

by P_1 in A. The reactions in A and B, produced by P_1 are, therefore, P_1' and P_1'', respectively, determined by the resolution of P_1 in the parallelogram indicated.

In like manner the reactions produced by P_2 are found to be P_2'' and P_2', acting in A and B respectively. The resultant reactions in A and B are then R_1 and R_2 respectively, R_1 being the resultant of P_1' and P_2'' and R_2 the resultant of P_1'' and P_2'.

If the forces R_1, R_2, P_1 and P_2 are combined in a force polygon, Fig. 1 *b*, and the equilibrium polygon $A\ e_1\ O\ e_2\ B$ is drawn, it will be seen that the reactions Q are equal and opposite, and that the line of action of $e_1\ O\ e_2$ is a straight line passing through the hinge O. This equilibrium polygon is called the line of thrust for the given forces.

The horizontal components of Q_1, Q_2, R_1 and R_2 are all equal to H, (see Fig. 1 *b*), which is called the horizontal thrust, and is constant for any point of the arch. The vertical components of R_1 and R_2 are the vertical reactions in A and B respectively, and the vertical component V of Q represents the shear at the crown O. Only these horizontal and vertical components of the reactions in A, O and B will be considered in the following, and will be called "the reactions."

The general expressions for the reactions of a three-hinged arch will now be found.

Assume, in Fig. 2, the segment $O\,B$ removed and a force R, resolved into components H (horizontal) and V (vertical), replacing the action of $O\,B$ on $O\,A$. The moment equation about A is

$$P_1 d_1 - Hf + \frac{Vl}{2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Similarly assume, in Fig. 3, the segment $O\,A$ removed and the

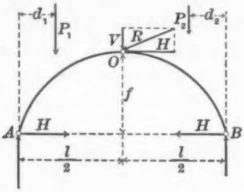

FIG. 2.

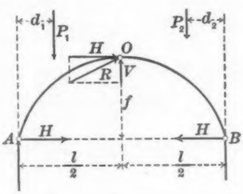

FIG. 3.

equilibrium of segment $O\,B$ preserved by the force R (equal and opposite to R in Fig. 2) resolved into components H and V as before. The moment equation about B is

$$- P_2 d_2 + Hf + \frac{Vl}{2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

evaluating H and V from (1) and (2)

$$H = \frac{1}{2f}\,(P_2 d_2 + P_1 d_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

$$\text{and } V = \frac{1}{l}\,(P_2 d_2 - P_1 d_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

The vertical reactions are obtained as follows from the equations for shear. With reference to Figs. 2 and 3,

$$A = P_1 + V = P_1 + \frac{1}{l}\,(P_2 d_2 - P_1 d_1) = \frac{P_1\,(l - d_1) + P_2 d_2}{l} \ldots \text{(5)}$$

$$B = P_2 - V = P_2 - \frac{1}{l}\,(P_2 d_2 - P_1 d_1) = \frac{P_2\,(l - d_2) + P_1 d_1}{l} \ldots \text{(6)}$$

From (5) and (6) it will be seen that the reactions A and B are identical with the reactions of a simple beam supported at A and B.

The value of R obtained from either Fig. 2 or Fig. 3 is

$$R = \sqrt{H^2 + V^2} = \sqrt{H^2 + S^2} \dots\dots\dots\dots (7)$$

which expression holds good for the resultant thrust at any point of the line of thrust, S being the shear at the point in question.

It follows from (3), that all forces acting on the arch, at points between A and B, affect H positively, while this is not true of the reactions A, B and V.

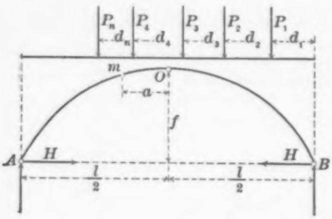

FIG. 4.

(*b.*) *Reactions Resulting from a Train of Concentrated Loads, Coming on the Span from the Right-Hand Abutment.*—The laws indicated by equations (3), (4), (5) and (6) are applied in deducing the expressions for the general case of loading, Fig. 4, as follows:

$$H = \frac{1}{2f}\left[\Sigma_o^B \; P\,d + \Sigma_o^A \; P\,(l - d)\right] \dots\dots\dots\dots (8)$$

$$V = \frac{1}{l}\left[\Sigma_o^B \; P\,d - \Sigma_o^A \; P\,(l - d)\right] \dots\dots\dots\dots (9)$$

$$A = \frac{1}{l}\,\Sigma_A^B \; P\,d. \dots\dots\dots\dots (10)$$

$$B = \frac{1}{l}\,\Sigma_A^B \; P\,(l - d) \dots\dots\dots\dots (11)$$

Also, the shear at any point m, distant a from the crown O is found as for a beam of span $A\,B = l$ and is

$$S = A - \Sigma_m^A \, P = \frac{1}{l}\,\Sigma_A^B \, P\,d - \Sigma_m^A \, P \dots\dots\dots\dots (12)$$

In the above equations the expression $\Sigma_o^B \; P\,d$ is used to indicate the sum of the products $P\,d$ for all loads acting between the points O and B. Other expressions of summations are to be interpreted accordingly.

(c.) *Reactions Resulting from a Uniform Live Load p per Unit of Length, Coming on the Span from the Right-Hand Abutment and Extending over the Span to a Distance e to the Left of the Crown O.*—The following equations are obtained analogous to equations (8), (9), (10) and (11):

$$H = \frac{1}{2f}\left[\frac{pl^2}{8} + \frac{pe(l-e)}{2}\right] \quad \text{.......................(13)}$$

$$V = \frac{1}{l}\left[\frac{pl^2}{8} - \frac{pe(l-e)}{2}\right] \quad \text{.......................(14)}$$

$$A = \frac{p}{2l}\left(\frac{l}{2} + e\right)^2 \quad \text{............................(15)}$$

$$B = p\left(\frac{l}{2} + e\right) - A = \frac{p}{2l}\left(\frac{3l^2}{4} + el - e^2\right) \quad \text{..(16)}$$

(*d.*) *Symmetrical Loading.*—For equal loads placed symmetrically with respect to the crown, or for symmetrical loading, equations (3), (4), (5) and (6), also equations (8), (9), (10) and (11), give the following special values:

$$H = \frac{\Sigma_0^A P d}{f}, \qquad V = 0 \qquad \text{and } A = B = \Sigma_0^A P.$$

For uniformly distributed load over the entire span, $e = \frac{l}{2}$ and equations (13), (14), (15) and (16) give

$$H = \frac{pl^2}{8f}, \qquad V = 0 \qquad \text{and } A = B = \frac{pl}{2}.$$

As only symmetrically shaped arches are to be treated in the following, the analysis will be confined to only the half span.

II.—Position of a Moving Load for Maximum Stresses.

For any arch of center line $A\ O\ B$ and hinged at A, O and B, Fig. 5, the reaction at B, resulting from any load P to the left of O, will have the direction $C\ O\ B$ and the reaction at A will pass through A and C. The reaction A will then produce a moment about any point

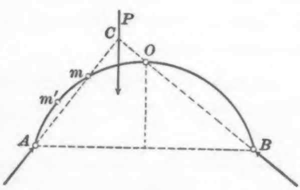

FIG. 5.

m' of the arch center line, which will be negative or positive accordingly as this point m' is above or below the line $A\ C$, and this moment will become zero for a point m on the line $A\ C$. Hence the vertical through C is the dividing line of loads or load divide for positive and negative influences on the moment about m, which point is the intersection of the line $A\ C$ with the center line $A\ O\ B$, though more properly the line of thrust.

Hence, a system of loads covering the span from the right abutment B up to C, will produce maximum compression in the fibers of

the intrados and maximum tension (if any) in the fibers of the extrados for the arch section at m. Also, a system of loads covering the span from the abutment at A up to C will produce maximum compression in the fibers of the extrados and maximum tension (if any) in the fibers of the intrados for this same arch section at m.

Since the resultant thrust in masonry arches is usually confined to the middle third of the arch ring (see Section IV) and accordingly this line of thrust is very nearly normal to the voussoir joints or *radii vectori*, it follows that the shear component of this thrust is necessarily small and any discussion with regard to loading for maximum and minimum shears is considered superfluous, especially as the unit stresses are somewhat liberally chosen, owing to the rather uncertain properties of masonry.

III.—Combined Action of Dead and Live Loads.

(*a.*) *Uniformly Distributed Live Load and Symmetrical Dead Load.*

1. Case of loading for maximum compression in the intrados for any point m of the left half of span: This case of loading will also give the minimum stress in the extrados at m. In accordance with the preceding, the live load must in this case cover the span from the right-hand abutment at D up to the load divide C for the point m, Fig. 6. The following definitions of terms will be adopted and retained throughout this work.

Let m be the point of application of the resultant thrust R of the external forces on any voussoir joint or radial section $m\,n$ under consideration.

Let y and a be the co-ordinates of the point m, referred to the rectangular axes $O\,E$ and $O\,n$ with origin at O.

$\Sigma_0^a\, q =$ resultant dead load of that portion of the span between O and m.

$b =$ distance from O to line of action of $\Sigma_0^a q$.

$\Sigma^{\frac{l}{2}}\, q =$ resultant dead load for the half span $O\,A$.

$s =$ distance from A to the line of action of $\Sigma_0^{\frac{l}{2}}\, q$.

$p =$ live load per unit of length, covering the span from D to C.

e = distance from crown O to the load divide C to the left of O.

H = horizontal thrust from total dead load A to B and live load from D to C.

V = shear at O due to live load D to C. The dead load being symmetrical does not affect V.

S = total shear at m from dead and live loads as assumed for H.

R = resultant thrust at m = resultant of H and S. This resultant passes through m and hence its moment about m must be zero, likewise the sum of the moments of the external forces about m must be zero.

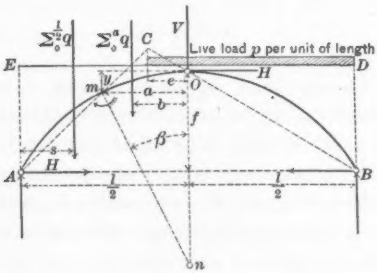

FIG. 6.

Since the dead load is not uniformly distributed in the case of an arch, the quantities, $\Sigma_o^1 q$ for successive positions of m are not proportional and must be obtained by summing the q's. The entire arch is divided into vertical segments and the weight q, of each, is determined, as is also the distance, r, from the crown, O, to the center of gravity of each q. The distances, b, are obtained by dividing the sums of the moments, $q\,r$, by the sums of the corresponding q's, thus

$$b = \frac{\Sigma_o^a q\, r}{\Sigma_o^a q}.$$

The moment equation of external forces about m is

$$(a - b)\, \Sigma_o^a q + a\, V + e\, p\left(a - \frac{e}{2}\right) - Hy = 0$$

which when solved for y gives

$$\left.\begin{aligned}
y &= \frac{(a-b)\,\Sigma_o^a\, q + a\,V + e\,p\left(a - \frac{e}{2}\right)}{H} \\
&\text{wherein, from equations (13) and (14),} \\
V &= \frac{1}{l}\left[\frac{p\,l^2}{8} - p\,e\frac{(l-e)}{2}\right] \\
&\text{and} \\
H &= \frac{s}{f}\,\Sigma_o^{\frac{l}{2}}\, q + \frac{1}{2f}\left[\frac{p\,l^2}{8} + p\,e\frac{(l-e)}{2}\right]
\end{aligned}\right\} \quad \dots\dots\dots\dots\dots (17)$$

Also, by analogy, with equations (12) and (15),

$$S = A - \Sigma_a^{\frac{l}{2}}\, q = \frac{p}{2l}\left(\frac{l}{2}+e\right)^2 + \Sigma_o^{\frac{l}{2}}\, q - \Sigma_a^{\frac{l}{2}}\, q = \frac{p}{2l}\left(\frac{l}{2}+e\right)^2 + \Sigma_o^a q \quad \dots\dots\dots (18)$$

Now, since for any point, m, whose absissa, a, is known or assumed, the ordinate, y, can be found from (17), the locus of m, for all points from A to O, is fully determined and can be drawn. This locus represents the extreme positions of all possible lines of thrust resulting from the combined action of dead load and moving live load.

The loci of maximum and minimum effects give at once the data for obtaining the thickness of arch ring for every assumed point, m (see Section IV).

2. Case of loading for maximum compression in the extrados for any point m of the left half of span. The same loading will also give the minimum stress in the intrados at m. In this case the live load must cover the span from the left hand abutment E up to the load divide C, Fig. 7.

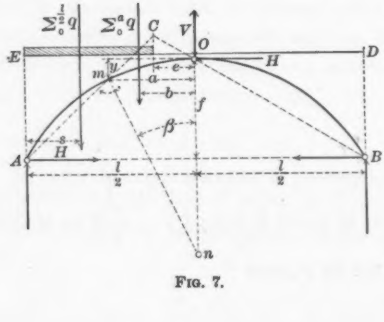

FIG. 7.

Retaining the previous notation, the moment equation for the point m is:

$$(a-b)\ \Sigma_o^a\ q - a\ V + \frac{p}{2}(a-e)^2 - H\,y = 0$$

which gives for y the value

$$\left.\begin{array}{l} y = \dfrac{(a-b)\ \Sigma_o^a\ q - a\ V + \dfrac{p}{2}\ (a-e)^2}{H} \\ \text{in which} \\ V = \dfrac{p}{2\,l}\left(\dfrac{l}{2} - e\right)^2 \text{ and } H = \dfrac{s}{f}\,\Sigma_o^{\frac{l}{2}}\ q + \dfrac{p}{4f}\left(\dfrac{l}{2} - e\right)^2 \end{array}\right\} \ldots\ldots\ldots(19)$$

also

$$\left.\begin{array}{l} S = \Sigma_o^{\frac{l}{2}}\,q + \Sigma_o^a\,q + p\,(a-e) - B = \Sigma_o^a\ q + p\ (a-e) - \\ \qquad\qquad\qquad\qquad\qquad\qquad \dfrac{p}{2\,l}\left(\dfrac{l}{2} - e\right)^2 \end{array}\right\} \ldots\ (20)$$

3. Case of loading for the half span covered with uniformly distributed live load for any point m of the left half of span. In this instance the locus of y represents a line of thrust for the assumed case of loading. The condition imposed makes $e = 0$, hence for load extending from D to O, (17) and (18) give

$$y = \frac{(a-b)\ \Sigma_o^a\ q + \dfrac{a\,p\,l}{8}}{H} \ldots\ldots\ldots\ldots\ldots\ldots\ (21)$$

and

$$S = \Sigma_o^a\ q + \frac{p\,l}{8} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (22)$$

and for load extending from E to O, (19) and (20) give

$$y = \frac{(a-b)\ \Sigma_o^a\ q + \dfrac{p\,a^2}{2} - \dfrac{a\,p\,l}{8}}{H} \ldots\ldots\ldots(23)$$

and

$$S = \Sigma_o^a\ q + a\ p - \frac{p\,l}{8} \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

In equations (21) and (23) the value of H remains constant for any position of m and has the value

$$H = \frac{s}{f}\,\Sigma_o^{\frac{l}{2}}\,q + \frac{p\,l^2}{16\,f} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

This condition of loading was formerly applied as a case for maximum stresses at the quarter points; but as is readily seen from the above, this assumption gives values which are much too small.

4. Case of loading for maximum values of H and S, the entire span being symmetrically covered with uniformly distributed live load. For this condition of loading, the shear V at the crown O becomes zero, and the locus of m represents a line of thrust for the imposed loads.

The equation of moments about any point m on the line of thrust is

$$(a-b)\,\Sigma_o^a\,q + \frac{a^2 p}{2} - H\,y = 0\,;$$

which gives

$$\left.\begin{array}{l} y = \dfrac{(a-b)\,\Sigma_o^a\,q + \dfrac{a^2 p}{2};}{H} \\ \text{in which} \\ H = \dfrac{s}{f}\,\Sigma_o^{\frac{l}{2}}\,q + \dfrac{p\,l^2}{8f} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(26)$$

also

$$S = \Sigma_o^a\,q + a\,p \text{ and } S_{max} = \Sigma_o^{\frac{l}{2}}\,q + \frac{p\,l}{2} = A_{max}. \ldots\ldots(27)$$

(b.) Train of Concentrated Live Loads and Symmetrical Dead Load.

1. Case of loading for maximum compression in the intrados for any point m of the left half of span. Using the notation given under (a) and assuming a system of loads as in Fig. 8, extending over the span from D to the load divide C, the following equations are derived in the manner already indicated for (17) and (18). The same case of loading will also give the minimum stress in the extrados at m.

$$\left.\begin{array}{l} y = \dfrac{(a-b)\,\Sigma_o^a\,q + a\,V + \Sigma_o^e\,P\left(\dfrac{l}{2} + a - d\right)}{H} \\ \text{in which} \\ V = \dfrac{1}{l}\left[\Sigma_o^{-\frac{l}{2}}\,P\,d - \Sigma_o^e\,P\,(l-d)\right] \\ \text{and} \\ H = \dfrac{s}{f}\,\Sigma_o^{\frac{l}{2}}\,q + \dfrac{1}{2f}\left[\Sigma_o^{-\frac{l}{2}}\,P\,d + \Sigma_o^e\,P\,(l-d)\right] \end{array}\right\} \ldots(28)$$

also

$$S = A - \Sigma_a^{\frac{l}{2}}\,q = \frac{1}{l}\,\Sigma_e^{-\frac{l}{2}}\,P\,d + \Sigma_o^{\frac{l}{2}}\,q - \Sigma_a^{\frac{l}{2}}\,q = \Sigma_o^a\,q + \frac{1}{\ }\,\Sigma_e^{-\frac{l}{2}}\,P\,d \;(29)$$

ERRATUM.

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Page 44, last line: The last function of equation (29) should read

$$\frac{1}{l}\,\Sigma^{-\frac{l}{2}}_{e}\,P\,d.$$

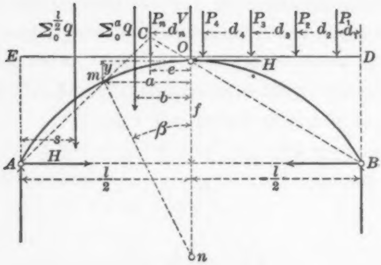

FIG. 8.

It will be seen from the value of V in (28) that a load P falling exactly at O will neither increase nor decrease the value of V.

2. Case of loading for maximum compression in the extrados for any point m of the left half of span. The same loading will also give the minimum stress in the intrados at m. The loads are assumed to cover the span from E to the load divide C, and the distances d are measured as before from the abutment at D. Notation as before, Fig. 8.

$$\left.\begin{array}{l} y = \dfrac{(a-b)\,\Sigma_o^a\, q - a\,V + \Sigma_e^a\, P\left(\dfrac{l}{2}+a-d\right)}{H} \\ \text{in which} \\ \qquad V = \dfrac{1}{l}\,\Sigma_e^{\frac{l}{2}}\, P\,(l-d) \\ \text{and} \\ \qquad H = \dfrac{s}{f}\,\Sigma_o^{\frac{l}{2}}\, q + \dfrac{1}{2f}\,\Sigma_e^{\frac{l}{2}}\, P\,(l-d) \end{array}\right\} \cdots\cdots (30)$$

also

$$\left.\begin{array}{l} S = \Sigma_o^{\frac{l}{2}}\, q + \Sigma_o^a\, q - B + \Sigma_e^a\, P = \Sigma_o^a\, q + \Sigma_e^a\, P \\ \qquad\qquad\qquad\qquad\qquad - \dfrac{1}{l}\,\Sigma_e^{\frac{l}{2}}\, P\,(l-d) \end{array}\right\} \cdots\cdots\cdots\cdots (31)$$

In order to obtain the greatest load effects in this and the previous cases, the heaviest loads should be placed near O in case 1, and at C in case 2.

(c.) *Loading as Under* (*a*) *Combined with a Concentrated Live Load W.*

1. Case of loading for maximum compression in the intrados for any point m of the left half of span, giving also the minimum stress in the extrados at m. Since a single concentrated load exerts its

maximum positive influence on y when applied just to the right of the crown O (as may be seen from equations (28)), the load W will be assumed to act at a unit distance to the right of O for any point m of the left half of span. Hence equations (17) and (18) will apply here when the effect of W is introduced (see Fig. 6).

$$\left.\begin{aligned} y &= \frac{(a-b)\,\Sigma_o^a q + a\,V + e\,p\left(a - \frac{e}{2}\right)}{H} \\ &\text{in which} \\ V &= \frac{1}{l}\left[\frac{p\,l^2}{8} + W\left(\frac{l}{2} - 1\right) - \frac{e\,p}{2}\,(l-e)\right] \\ &\text{and} \\ H &= \frac{s}{f}\,\Sigma_o^{\frac{l}{2}} q + \frac{1}{2f}\left[\frac{p\,l^2}{8} + W\left(\frac{l}{2} - 1\right) + \frac{e\,p}{2}\,(l-e)\right] \end{aligned}\right\} \cdots (32)$$

Also

$$S = \Sigma_o^a q + \frac{p}{2l}\left(\frac{l}{2} + e\right)^2 + \frac{W}{l}\left(\frac{l}{2} - 1\right) \cdots\cdots\cdots (33)$$

2. Case of loading for maximum compression in the extrados for any point m of the left half of span, producing also the minimum stress in the intrados at m. Here W exerts its maximum influence on y when applied vertically over m, and equations (19) and (20) are modified as follows. See, also, Fig. 7.

$$\left.\begin{aligned} y &= \frac{(a-b)\,\Sigma_o^a q - a\,V + \frac{p}{2}\,(a-e)^2}{H} \\ &\text{in which} \\ V &= \frac{1}{l}\left[\frac{p}{2}\left(\frac{l}{2} - e\right)^2 + W\left(\frac{l}{2} - a\right)\right] \\ &\text{and} \\ H &= \frac{s}{f}\,\Sigma_o^{\frac{l}{2}} q + \frac{1}{2f}\left[\frac{p}{2}\left(\frac{l}{2} - e\right)^2 + W\left(\frac{l}{2} - a\right)\right] \end{aligned}\right\} \cdots\cdots (34)$$

also

$$S = \Sigma_o^a q + p\,(a-e) - \frac{1}{l}\left[\frac{p}{2}\left(\frac{l}{2} - e\right)^2 + W\left(\frac{l}{2} - a\right)\right] \cdots (35)$$

IV.—Conditions of Stress on a Radial Section of an Arch.

(*a*) *The Resultant Normal Thrust on the Section.*—The resultant thrust R at any point m of a linear arch $A\,O$, Fig. 9, is obtained from equation (7), as $R = \sqrt{H^2 + S^2}$, in which H is the horizontal thrust and S the vertical shear for this point. But as R is not usually normal to the radial section $m\,n$, it will have components perpendicular to and parallel with this section.

The resultant R is resolved into N (the normal thrust) and T (the tangential thrust) respectively perpendicular to and parallel with the radial section $m\,n$, which section is represented by the radius vector of the curve $A\,O$ at the point m and makes the angle β with the vertical.

The values of N and T in terms of H and S are now found without involving R in the result.

The forces S, H, R, N and T are shown in their proper relationship in Fig. 9, from which the following equations are obtained:

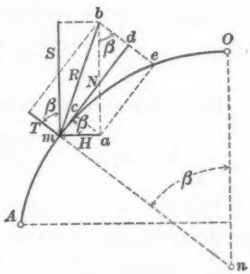

FIG. 9.

The angle $a\,m\,d$ = the angle $a\,b\,d$ = the angle $O\,n\,m = \beta$, also

$$a\,b = S, \text{ and } b\,d = T$$

also

$$a\,e = c\,d = a\,b \sin\beta = S \sin\beta$$

and

$$m\,c = m\,a \cos\beta = H \cos\beta$$

hence

$$N = m\,d = m\,c + c\,d = H \cos\beta + S \sin\beta \quad \text{.................(36)}$$

and

$$T = b\,d = b\,e - e\,d = S \cos\beta - H \sin\beta \quad \text{.................(37)}$$

However, the tangential force T rarely becomes sufficiently large to require any consideration, especially when an arch is so designed that no tensile stresses will ever occur, thereby confining the thrust R to the middle third of the arch-ring and reducing the angle between R and N to a very small quantity. Also the high factors of safety (6 to 10) used in masonry arches would not warrant the consideration of so small a factor as T.

(b) *The Stresses on the Section.—1. Analytical Solution.* In Fig. 10, let $A\,B = D$ represent the thickness of the arch on a radial section $m\,n$; also let g be the middle point of $A\,B$, and the point of application of the resultant thrust N, distant w from g. Required, the manner in which N is distributed over the section and the intensities of the unit stresses k_1 and k_2 on the extreme elements of the arch-ring.

If N is given for an arch of unit breadth then k_0, k_1 and k_2 will be unit stresses.

For N acting in g, it is evident that the stress is uniformly distributed over $A\,B$ and $k_0 = k_1 = k_2 = \frac{N}{D}$. For N acting at a distance

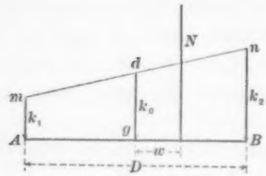

FIG. 10.

w to the right of g, k_1 will be less than k_0, and k_2 will be greater than k_0, each by an amount $f = \frac{M\,y}{I}$ representing the effect of the moment $N\,w$.

Therefore,

$$k_1 = k_0 - \frac{M\,y}{I} \text{ and } k_2 = k_0 + \frac{M\,y}{I} \quad ..(38)$$

in which

$$y = \frac{D}{2},\ I = \frac{b\,h^3}{12} = \frac{D^3}{12} \text{ for } b = 1 \text{ and } M = N w = w\,Dk_0$$

By substituting these values in 38 and reducing

$$k_1 = k_0 \left(1 - \frac{6\,w}{D}\right) \text{ and } k_2 = k_0 \left(1 + \frac{6\,w}{D}\right) \quad \ldots\ldots\ldots\ (39)$$

The line $m\ d\ n$ represents the manner of distribution of stress produced by the resultant N on the section $A\ B$.

2. *Graphical Solution.* Equations (39) may be written thus, $k_1 = \frac{6\,k_0}{D}\left(\frac{D}{6} - w\right)$ and $k_2 = \frac{6\,k_0}{D}\left(\frac{D}{6} + w\right)$, in which form they may be represented graphically.

In Fig. 11, draw to scale of forces $k_0 = \frac{N}{D}$ at g, perpendicular to $A\ B$, and lay off distances $e\ g$ and $b\,g = \frac{D}{6}$ to the right and left from g. Draw $e\ d$ and prolong same to intersect N in h, also draw $d\ b$ intersecting N in i. Then will $i\ t = A\ m = k_1$, and $t\ h = B\ n = k_2$, and $m\ d\ n$ will represent the law of variation of stress over the section $A\ B$.

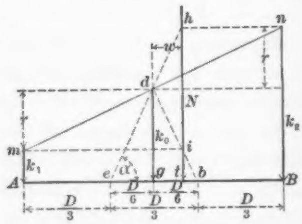

FIG. 11.

$$\text{For } k_1 = k_0 - r = t\,b \tan\alpha = \frac{6\,k_0}{D}\,t\,b = \frac{6\,k_0}{D}\left(\frac{D}{6} - w\right)$$

$$\text{and } k_2 = k_0 + r = e\,t \tan\alpha = \frac{6\,k_0}{D}\,e\,t = \frac{6\,k_0}{D}\left(\frac{D}{6} + w\right).$$

In equations (39) when $w = 0$, $k_1 = k_2 = k_0$, and when $w = \frac{D}{6}$, $k_1 = 0$ and $k_2 = 2\,k_0$. When $w > \frac{D}{6}$, k_1 becomes negative or tensile. Hence

to avoid tensile stresses on any section $A\,B$, the resultant normal thrust N must have its point of application t within the middle third $e\,b$ of said section.

(c) *Thickness of Arch Ring.*—Given the direction, amount and point of application of the normal thrusts N_e and N_i, obtained from the loading for maximum compression in the extrados and intrados respectively, for a radial section $A\,B$, Fig. 12, to find the thickness D of the arch ring which must be provided so that a certain assigned unit stress k on the extreme elements of the ring shall never be exceeded.

While N_e and N_i can never occur simultaneously, the center line of the arch must be so placed, with respect to these thrusts, and a minimum value of D, that the above conditions may be complied with.

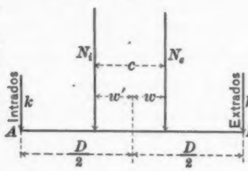

FIG. 12.

The dimension c in Fig. 12 is obtained from the difference between maximum and minimum y, which vertical difference is projected on the radial section $A\,B$ by multiplying with $\cos\beta$; hence $c = \triangle y \cos\beta$ is a known quantity, and w and w' are to be found, likewise D.

The values k_o in equations (39) for the two thrusts are $k_o = \frac{N_e}{D}$ and $k'_o = \frac{N_i}{D}$, and from the figure $w' = c - w$. These values substituted in the second equation (39) give the two following values of $k_2 = k$, the given allowable compressive stress:

$$k = \frac{N_e}{D}\left(1 + \frac{6\,w}{D}\right) \text{ and } k = \frac{N_i}{D}\left(1 + \frac{6\,(c - w)}{D}\right)$$

These equations when solved for D and w give the dimensions sought.

$$D = \frac{N_i}{k\left(1 + \frac{N_i}{N_e}\right)} + \sqrt{\frac{6\,c}{k}\left[\frac{N_i}{1 + \frac{N_i}{N_e}}\right] + \left[\frac{N_i}{k\left(1 + \frac{N_i}{N_e}\right)}\right]^2} \dots \quad (40)$$

$$w = \frac{k\,D^2}{6\,N_e} - \frac{D}{6} \dots\dots\dots\dots\dots\dots \quad (41)$$

The ordinate of the arch centerline for the section $A\,B$ is also found from w, and the ordinate to the point of application of N_e which

may be called y_{min}. Calling the ordinate of the center line y_c, and the inclination of $A\ B = \beta$ from the vertical,

$$y_c = \frac{w}{\cos\ \beta} + y_{min} \qquad (42)$$

The values of k_1 for the above thrusts and dimensions may now be found from the first of equations (39).

(*d*) *Tensile Stresses.*—It may occur that after having found D from (40) and k_1 from (39), the latter value may be negative or tensile. Should this tensile stress be in excess of the allowable stress then the thickness D must be further increased, or if the material of the arch be concrete the excessive tension may be taken up by the insertion of wire netting or iron rods. The method of computing the area of metal required is given in the following:

In Fig. 13, let $A\ B$ be the radial arch section k_1 a tensile stress on the extreme element at A and k_2, the corresponding compressive stress at B.

$a\,b$ is a steel wire placed at a distance v from the axis of the arch. Let a be the area of the steel required for an arch ring of unit breadth and $f =$ the allowable unit stress for steel.

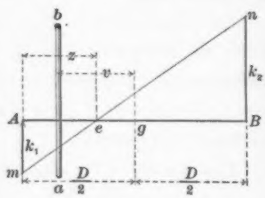

FIG. 13.

The total tensile stress on the arch ring (for unit breadth) will be represented by the area $A\ m\ e = \frac{k_1\, z}{2} = u'$.

From the similar triangles $A\ m\ e$ and $B\ e\ n$

$$\frac{z}{D - z} = \frac{k_1}{k_2},\ \text{or}\ z = \frac{k_1\ D}{k_1 + k_2};\ \text{making}\ u' = \frac{k_1^2\ D}{2\ (k_1 + k_2)} \qquad (43)$$

The point of application of u' is the center of gravity of the triangle $A\ m\ e$ which is distant from g by an amount $\frac{D}{2} - \frac{z}{3}$. Hence the moment of u' about g is $M = u' \left(\frac{D}{2} - \frac{z}{3}\right)$ which, when taken up by the steel rod, produces therein the force

$$u = \frac{u' \left(\frac{D}{2} - \frac{z}{3}\right)}{v} \qquad (44$$

But $a = \frac{u}{f}$; hence from (43) and (44) the value of a is found as

$$a = \frac{k_1^2 \, D\left(\frac{D}{2} - \frac{k_1 \, D}{3\,(k_1 + k_2)}\right)}{2\,f\,v\,(k_1 + k_2)} \qquad \text{(4}$$

In equation (45) k_1 is tensile stress and k_2 is compressive stress, and both enter into the equation without regard to sign of stress.

V.—Deformations of the Arch Ring.

(a) General Considerations; Changes in Length of the Arch Ring.—An arch ring, when under the influence of stress and changes in temperature, is subject to elastic deformations. The compressibility of masonry by stress and the shrinkage caused by the setting process of mortar and concrete, bring about a permanent shortening in the arch ring during construction and test loading.

The resultant effect of these several shortenings and temperature changes produces a deformation of the arch ring which must be provided for in the construction, to prevent a deflection below the normal, which, if it occurred, would materially increase the horizontal thrust on the abutments. This superelevation, which must be given to the arch ring in order that it may attain its proper rise when completed, is called camber.

Besides the above deformation, which is partly elastic and partly permanent, provision must also be made for the settlement in the falsework caused by the weight of the arch ring up to the time of closing, when the latter becomes self-sustaining. This part of the problem is here omitted as depending entirely on the nature of the falsework and local building conditions.

The change in length of the arch ring resulting from stress, temperature, compressibility of material and shrinkage of masonry will now be found.

The normal arch thrust N may be found, for any combination of dead and live loads, from the preceding equations, and this thrust increases from the crown toward the haunches. The cross-section of the arch ring is also a variable quantity. Hence it is necessary to divide the arch ring into sections over which the cross-section and normal thrust may be considered constant. The sum of the increments of change of these sections gives the change in length of the arch ring.

Let L = length of a section of arch, over which the area and thrust are assumed constant.

$\triangle L$ = increase in the length L for any assigned reasons. Decrease is negative.

$\delta = \Sigma_0^{\frac{l}{2}} \triangle L$ = sum of the changes $\triangle L$ from the crown to the abutment.

N = normal thrust acting through the length L.

F = average cross-section of the section of length L.

E = modulus of elasticity of the material for the working stress $\frac{N}{F}$.

E' = modulus of permanent set of the material for the working stress $\frac{N}{F}$

t = a rise, and $-t$ = a fall, in temperature, in degrees Centigrade.

α = coefficient of expansion for 1° Centigrade.

ε = coefficient of shrinkage from setting of mortar or concrete in air.

then
$$\triangle L = -\frac{N L}{E F} - \frac{N L}{E' F} - \varepsilon L + \alpha t L$$
$$= -\frac{N L}{F}\left(\frac{1}{E} + \frac{1}{E'}\right) - L\left(\varepsilon - \alpha t\right) \text{ .... } (46)$$

and
$$\delta = -\left(\frac{1}{E} + \frac{1}{E'}\right) \Sigma_0^{\frac{l}{2}} \frac{N L}{F} - \left(\varepsilon - \alpha t\right) \Sigma_0^{\frac{l}{2}} L \text{ ...... } (47)$$

The elastic change in length of an arch ring in a completed structure no longer subject to increased permanent set is

$$\delta' = -\frac{1}{E} \Sigma_0^{\frac{l}{2}} \frac{N L}{F} + \alpha t \Sigma_0^{\frac{l}{2}} L \text{ .... } (48)$$

(*b*) *Deformation; Analytical Solution.*—To find the deflection at the crown of an arch ring, which latter has shortened an amount δ, as found from (47) or (48).

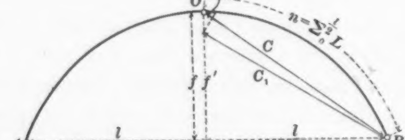

FIG. 14.

Let C, Fig. 14, be the chord from crown to abutment, correspond-

ing to the half span $\frac{l}{2}$ and rise f; C_1 = the length of this chord after the half arch ring has shortened an amount δ and the rise has diminished to f^1, the span remaining unchanged; also, $n = \Sigma_0^{\frac{l}{2}} L$ = the length of the center line of the arch ring between the crown and the abutment.

A shortening δ in the length n will produce a shortening $\frac{C}{n}\,\delta$ in the chord C, making $C_1 = C - \frac{C}{n}\,\delta$.

From the figure $C = \sqrt{f^2 + \left(\frac{l}{2}\right)^2}$, which, substituted in the previous equation, gives $C_1 = \left(1 - \frac{\delta}{n}\right)\sqrt{f^2 + \left(\frac{l}{2}\right)^2}$ ...... (49)

and
$$\triangle f = f - f^1 = f - \sqrt{C_1^2 - \left(\frac{l}{2}\right)^2} \quad \ldots\ldots (50)$$

The value $\triangle f$, which represents the deflection at the crown, may then be found from equations (49) and (50).

While the value $\frac{C}{n}\,\delta$ is not exactly correct, yet the approximation is so close that the resulting equations give values entirely within the knowable accuracy even for long spans.

As will be noticed, this solution applies to arches of any shape not necessarily circular, but is most accurate for circular arches.

(c) *Deformation; Graphical Solution.*—The general case for any condition of unsymmetrical loading and displacements of abutments can best be solved by the graphical method.* Only the application of the method is here given, without repetition of its derivation, for which see above-named article.

In Fig. 15*a*, let $A\ O\ B$ represent the line of thrust of a three-hinged arch for any particular case of loading; L_1, L_2, L_3, etc., the sections into which the arch is divided; and $-\triangle 1$, $-\triangle 2$, $-\triangle 3$, etc., the contractions in the lengths L_1, L_2, L_3, etc., respectively, caused from any combination of conditions, as stress, shrinkage of masonry, temperature, etc.

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* See article "Distortion of a Framed Structure," by David Molitor, in *Journal of the Association of Engineering Societies*, Vol. xiii, p. 310.

In Fig. 15 *b*, draw in succession, from any point A', the contractions $-\triangle 1$, $-\triangle 2$, $-\triangle 3$, etc., respectively, parallel to the elements L_1, L_2, L_3, etc., of Fig. 15 *a*, and in the direction in which these contractions act relatively to the fixed point A. In the example, the $\triangle$'s being negative, the elements of the arch, Fig. 15 *a*, all move toward A, and hence the quantities $-\triangle 1$, $-\triangle 2$, $-\triangle 3$, etc., are applied downward from A'. The broken line $A'\ O'$ then represents the motion of the point O relatively to the point A, assuming that the arch elements move parallel to themselves. In like manner, the broken line $O'\ B'$, Fig. 15 *c*, represents the motion of the point O relatively to the point B.

However, the elements of the arch do not move parallel to themselves, but the half arch $A\ O$ revolves about A and the half arch $O\ B$ revolves about B until the point O, common to both halves, attains its new position. This revolution is performed by a second operation as follows:

In Fig. 15 *b*, draw $A'\ O''$ perpendicular to $A\ O$ and through A'; also through O' draw $O'\ r$ perpendicular to $A'\ O''$, and in Fig. 15*c*, through B' draw $B'\ O''$ perpendicular to $B\ O$, and through O' draw $O'\ s$ perpendicular to $B'\ O''$. Now transfer $O'\ s$ from Fig. 15 *c* to Fig. 15 *b*, parallel and equal to itself, and in Fig. 15 *b* draw $s\ O''$ perpendicular to $O'\ s$, and $O''\ O'$ will represent the direction and amount of the displacement of the point O. Also transfer $O'\ r$ from Fig. 15 *b* to Fig. 15*c*, and in Fig. 15 *c* draw $r\ O''$ perpendicular to $O'\ r$, and $O''\ O'$ will check in direction and amount the value $O''\ O'$ found in Fig. 15 *b*.

To find the displacements of the individual points 1, 2, 3, etc., draw a broken line A', $1''$, $2''$, $3''$, $4''$, O'' with its elements proportional and respectively perpendicular to the elements A, 1, 2, 3, 4, O. The lines $1''\ 1'$, $2''\ 2'$, etc., will represent the true displacements of the the points 1, 2, etc., respectively. The same construction applied to Fig. 15 *c* gives the displacements of the points of the half arch $O\ B$.

When the abutments are displaced by amounts $-\triangle A$ and $-\triangle B$ respectively, these displacements are embodied in the diagrams of Figs. 15, *b* and *c*, as indicated by the dotted construction, and the displacements are then measured from the dotted broken lines O_1'', A_1', and O_1'', B'_1, respectively, to the broken lines $O'\ A'$ and $O'\ B'$.

For symmetrical loading, where the displacements in the half arch $A\ O$ are exactly equal to those in the half arch $B\ O$, the line of thrust is symmetrical, and the deflection at the crown O must be vertical.

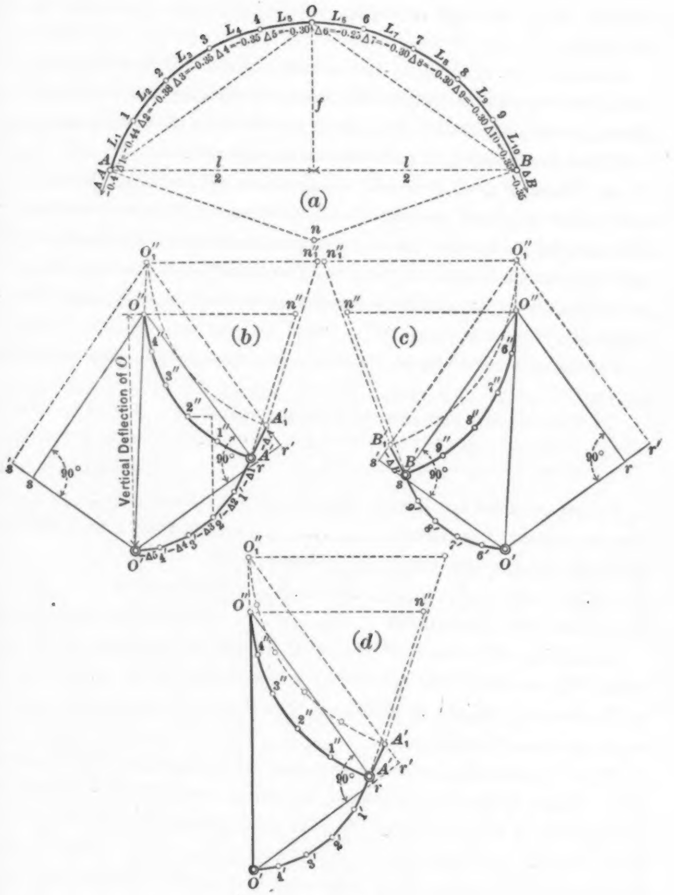


FIG. 15.

This extra condition makes it possible to dispense with one of the diagrams, and the solution becomes as shown in Fig. 15 *d*. The point O'' is then the intersection of $A'\ O''$ perpendicular to $A\ O$, and the vertical $O'\ O''$ through O'. This vertical becomes the deflection at the crown.

It should be noticed in the solution Figs. 15, that Fig. 15 *a* is drawn to a small scale, while the displacement diagrams *b*, *c* and *d* are drawn to natural scale. For small contractions these diagrams may be drawn to a scale of 10 to 100 times the natural.

(*d*) *Values of* ε, α, E *and* E' *in Equations* (*46*) *to* (*48*). While it is always best to make accurate determinations of the above values for the material to be used in any particular structure, especially when close agreement is desired, yet a brief summary of the meagre data on this subject may be useful as furnishing values for preliminary computations.*

Values of ε for German Portland cement mortars, sixteen weeks old.†

Mortar mixed 1 part cement to	0 part sand..	$\varepsilon = 0.0012$	to	0.0034
" 1 "	3 " ..	$\varepsilon = 0.0008$	"	0.0015
" 1 "	5 " ..	$\varepsilon = 0.0008$	"	0.0014

Values of α for one degree of temperature (Centigrade).

Cement mortar (Bruniceau) .... $\alpha = 0.00001$ to 0.000014

Portland cement concrete 1 : $2\frac{1}{2}$: 5 (Bauschinger) .... $\alpha = 0.0000088$

Stone and brick (Bruniceau) .... $\alpha = 0.0000053$ to 0.0000083

Regarding the values of E and E' it will be necessary to recite briefly the results of the interesting set of experiments made in 1894 by Professor C. Bach, of Stuttgart, which are of vital importance to the subject here treated.

These experiments showed that any concrete when subjected to stress would undergo a permanent set and an elastic deformation, the magnitudes of which become constant after several repetitions of the same stress. The number of repetitions necessary to produce constant deformations appeared to be a function of the breaking strength of the concrete, and of the intensity of the applied stress. The greater the ultimate strength, the less repetitions were required; and the greater the applied stress, the greater the number of repetitions.

* See article by David Molitor on "Properties of Concrete under Compressive Stress," *Jour. Assoc. Eng. Soc.*, May, 1896.

† See report of Committee on Compressive Strength of Cements, etc. *Transactions* Am. Soc. C. E., Vol. xv, p. 717.

However, for each kind of concrete a maximum stress was reached (about 0.7 of breaking stress) for which seven to eight repetitions would still continue to increase the deformations. Presumably more repetitions would have restored constancy, but this point may be regarded as a natural limit of allowable stress, though there does not appear to be a definite limit of elasticity.

The tests were made on cylindrical samples of concrete, 1 m. long and 25 cm. in diameter, mixed 1 part Portland cement to 2½ parts sand to 5 parts broken limestone, and 1 part cement to 3 sand to 6 stone, age two to three months. The breaking strength was in every case determined from the cylindrical samples. The strength of the same concrete developed by cubic samples would have been about one and one-half times the strength obtained from the long cylinders.

The following tables give values of E and E' for various values of ultimate strength and applied loads. These values are given in metric atmospheres.

VALUES OF E IN THOUSANDS (Three ciphers should be added to tabulated values).

ULTIMATE STRENGTH IN ATMOSPHERES.		APPLIED LOADS IN ATMOSPHERES, OR KGR. PER SQ. CM.							
Found from cylinders.	Estimated for cubes.	8	10	15	20	25	30	35	40
60	90	228	223	214	204	193	181	171	163
80	120	262	258	246	234	222	210	200	191
100	150	296	290	278	266	255	244	235	224
120	180	320	314	298	286	276	267	257	247
140	210	340	333	317	303	291	280	270	260

VALUES OF E' IN THOUSANDS (Three ciphers should be added to tabulated values).

ULTIMATE STRENGTH IN ATMOSPHERES.		APPLIED LOADS IN ATMOSPHERES, OR KGR. PER SQ. CM.							
Found from cylinders.	Estimated for cubes.	8	10	15	20	25	30	35	40
60	90	1 230	1 180	1 030	910	810	720	630	545
80	120	1 800	1 710	1 500	1 320	1 180	1 020	870	755
100	150	2 500	2 400	2 100	1 880	1 680	1 460	1 300	1 160
120	180	3 400	3 210	2 800	2 500	2 220	2 010	1 820	1 650
140	210	4 400	4 190	3 600	3 160	2 780	2 500	2 280	2 100

The important conclusions arrived at by Tourtay,* regarding compressive properties of masonry, are here given.

* *Annales des Ponts et Chaussées*, 1885, ii, p. 15.

1. The ultimate strength of small cubes of cement mortar is considerably less than the compressive strength of blocks of masonry made with the same mortar.

2. The pressure which crushes the masonry is an inverse function of the thickness of the mortar joint.

3. Stone plates laid loosely upon each other have a much smaller compressive strength than solid cubes.

4. The same stone plates, when cemented together with neat cement grout, possess the same compressive strength as do solid stone samples.

Practical Applications.

Solution of a Problem.

(*a*) *Introductory.*—To illustrate more clearly the method of arch construction as proposed in the foregoing, a problem is solved in sufficient detail to bring out the practical applications of the theory.

In designing metal bridges, the dead load is generally known, the more accurate for the forms most in use. This is not the case with masonry arches for reasons of insufficient experimental data. Also, a steel arch may be designed with any rational center line because tensile stresses can be provided for, while in masonry only compressive stresses are safely permissible; therefore, the shape of a masonry arch is of necessity a function of its loading.

However, with the dead load known, the three-hinged masonry arch can be analysed and dimensioned with the same degree of accuracy, consistent with the nature of the material, as is possible in steel. But, as the dead load can be obtained in no other way than by computation from assumed dimensions, the solution must be reached by successive approximations of dimensions until the loading resulting therefrom produces stresses in the arch which are not in excess of the allowable unit stresses. Much depends on the experience of the designer as to the rapidity and directness with which a solution may be obtained.

The method which it is believed will lead to a solution with a minimum of useless computation is illustrated in the following problem.

(*b*) *Statement of the Problem.*—Design a three-hinged concrete arch of 72-m. span between pin centers, rise, 10 m., width of driveway,

8 m., and footwalks 2 m. wide, on each side of the driveway. The profile of the site is given, and the abutments are to be founded on hard clay. The bridge is to carry an electric motor car weighing 20 000 kgr. (44 000 lbs.) on a track of 1.43 m. (4-ft. 8½-in.) gauge, and a uniformly distributed live load of 400 kgr. per $m.^2$ (82 lbs. per square foot).

The maximum allowable working stresses are 40 atm.* (568.8 lbs. per square inch) in compression, and 2 atm. (28.4 lbs. per square inch) in tension, for concrete composed of 1 part Portland cement to 2 parts sand to 3 parts crushed limestone. This concrete must attain the following compressive strengths on 20-cm. cubes: 220 atm. in 28 days, 350 atm. in 6 months, and 500 atm. in 2 years. The tensile strength of mortar composed of 1 part cement to 3 parts sand must be at least 20 atm. in 28 days.

(c) *Outline of the Method.* (See Plates I and II)—In this as in all bridge designs, the details of the roadway and its supports on the arch ring are first designed. By so doing the dead loads may be computed, involving the weight of the arch ring as the only variable subject to correction.

A diagram of the half span (see dimension diagram, Plate II) is then prepared and the roadway and its supporting columns drawn, to which is fitted an arch ring of seeming good proportions for the given span and rise. Herein, the experience of the designer is practically his only guide, but reference to a completed design or a solution once made will assist wonderfully in making close approximations. It will be seen from the side elevation, Plate I, that the center line (a three-center curve) is almost a complete circular arc, slightly flattened at the crown.

The concentrated dead loads, q, are now computed by using the dimensions scaled from the preliminary diagram. These values, together with values of a, r and e, are tabulated together with computed values of $q\,r$ (see Table No. 1).

Then assuming the maximum case of loading over the entire span, compute S_{max} from equation (27), H_{max} from equation (26) and $N_{max} = H_{max} \cos \beta + S_{max} \sin \beta$, from which may be found the required thickness of arch ring at the crown $D_o = \frac{H_{max}}{k}$ and at the abutments

* Stresses and pressures will be given throughout this problem in metric atmospheres equal to 1 kgr. per $cm.^2$ (14.22 lbs. per square inch).

$D_A = D_B = \frac{N_{max}}{k.}$. If it appears that the assumed dimensions differ widely from those computed, it is well to make a second approximation before proceeding further with the solution, though no correct idea can yet be formed of the required arch thickness at the quarter points. The arch thicknesses, at the crown and springing, should be chosen about 10% greater than found by computation, since these sections will be adjacent to the hinges and some allowance must be made for unequal distribution of pressure due to friction in the hinged bearings.

When the solution appears satisfactory to this stage, the assumed arch ring should be tested at two or three points between the crown and springing. The maximum thickness of the arch ring will occur at a distance of about 0.3 l from the crown. This and two other sections, distant about 0.15 l and 0.4 l from the crown, should be dimensioned, after which it will usually be possible to proceed to the final design and computation, unless the agreement with the preliminary design was not sufficiently close, in which case the process should be repeated by approximating a new arch ring, using the dimensions last obtained. In this latter case some allowance should be made for the probable effect of the new dead loads. The larger the ratio between dead and live loads, the less will be the divergence $\triangle y$ of maximum and minimum thrusts, though an increase in dead loads will also increase the resultant normal thrust N, on any section.

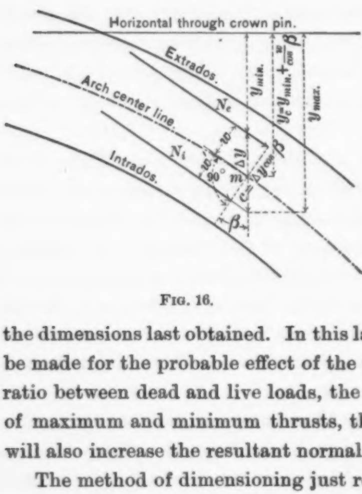
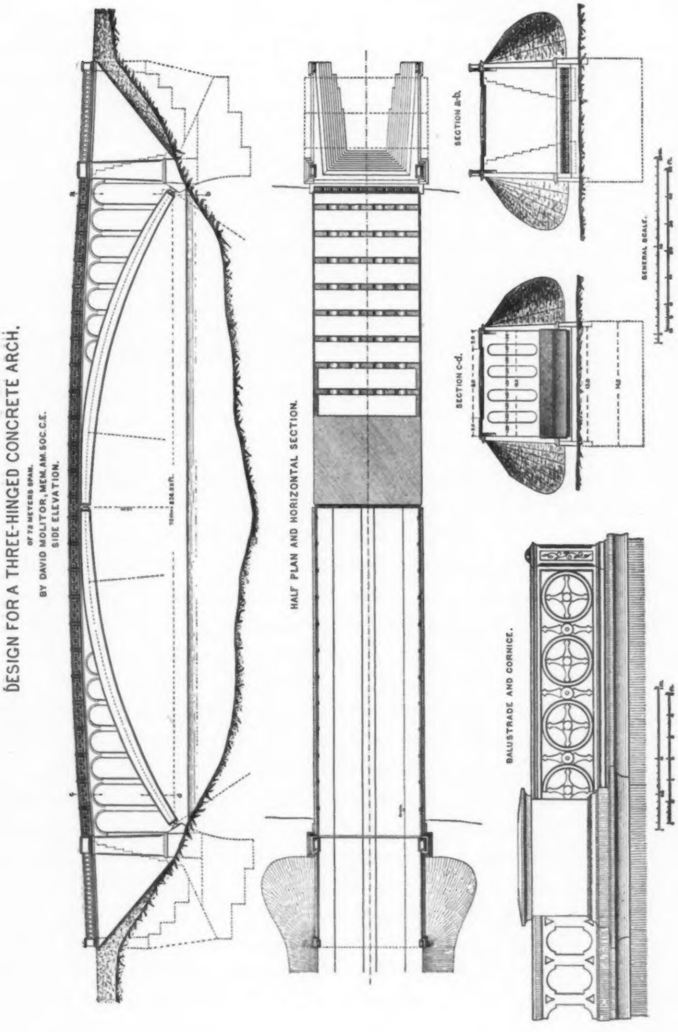


FIG. 16.

The method of dimensioning just referred to consists in computing maximum and minimum y's and the corresponding H's for each of the three points, using in the present problem equations (32) and (34). The differences of the y's give the $\triangle y$'s in Fig. 16. Then, finding S from equations (33) and (35) for each of the three points, compute N_e and N_i from equation (36). From equations (40), (41) and (42) the values D, w and y_c, respectively, may be found. The value of c in equation (40) is equal to $\triangle y \cos\beta$, as may also be seen in Fig. 16.

PLATE I.
TRANS. AM. SOC. CIV. ENGRS.
VOL. XL, No. 834.
MOLITOR ON THREE-HINGED MASONRY ARCHES.



All the factors for five points, including crown and springing, being now fully determined, the intermediate values can easily be interpolated in making the final dimension diagram.

The new diagram is now accurately drawn, and the design carefully computed, in the manner just outlined for the three test points, by application of the method to as many points as may be deemed necessary or desirable to fully determine the dimensions and stresses throughout the arch.

There is still one other point which is interesting to know and is illustrated in the final computation. The center line of the arch ring and the line of thrust for the case of one-half the uniformly distributed live load or $\frac{p}{2}$ acting over the entire span, are so nearly alike that no appreciable error would be introduced by accepting this line of thrust for the center line.

The above outline of the method to be followed is now applied to the problem in hand. In the following computations, the arch ring is assumed 1 m. wide, and the live load thus becomes 400 kgr. per meter, combined with a concentrated live load of $W = 10\ 000$ kgr.

(*d.*) *Design of the Roadway.*—From æsthetic considerations and for economic reasons, the roadway is designed to the parabolic curve whose equation is $-y = 0.001\,x^2$, making the same 1.3 m. higher at the crown than at the abutments (see Plate I).

The roadway is made up of concrete floor arches of 2.4 m. span, carried by small concrete piers resting on the arch ring. The cross-section of the driveway is the arc of a circle having a middle ordinate of 0.12 m., and is covered with 6 cm. of asphalt composition. The footwalks are sloped toward the bridge axis with a slope of 1 : 100, and are finished in cement mortar. The car track is placed in the center of the driveway and hard paving brick laid adjacent to the rails.

The horizontal thrust of the floor arches is taken up by steel rods 2.5 cm. in diameter, and spaced 17 cm. apart under the driveway and 75 cm. under the footwalks. Expansion joints are provided in the roadway at the crown and at the abutments, all as shown on Plate II. Drainage of the roadway is provided by cast-iron pipes placed in the first piers adjacent to the abutments, one for each gutter.

In accordance with this design the weight of the floor, complete, to the springing of the arches and between the pier centers, is estimated as follows:

Concrete for arch and portion of pier to springing, 0.57 m.^{3} at 2 300 kgr.	1 311 kgr.
Six steel rods, 2.5 cm. diam., 3 m. long.	71 "
Rails and I-beam.............	105 "
Six cm. asphalt composition at 2 200 kgr. per m.^{3}...	396 "
Total weight to be carried by one pier......	1 883 kgr.

The round number 1 900 kgr. was used.

The floor arches are designed to carry 10 000 kgr. evenly distributed over the half span (unsymmetrical load $q + p$, diagram Plate II), and the center line corresponds to the line of thrust following from 10 000 kgr. distributed over the entire span (loads $q + \frac{p}{2}$). The lines of thrust for these loads are drawn for arch horizontal, also for the inclined position between piers 7 and 8. The maximum horizontal thrust is 9 m.^{3} of concrete or 20 700 kgr. for an arch 1 m. wide. The maximum compression at the crown is 17 atm. and at the haunches and quarter points it is 36.8 atm. No tension occurs anywhere.

The longitudinal section investigated is at the curb of the driveway where the minimum thickness of arch (15 cm.) is possible. However, this portion of the roadway, according to the above computation, is amply strong to carry a 20-ton road roller. As a result of the crowning of the roadway the floor arches attain a thickness of 27 cm. in the axis of the bridge.

(*e*) *Design of the Arch Ring.*—A preliminary diagram similar to that shown on Plate II, is now drawn and the dead loads q are computed and tabulated, together with values of a, e, r and β, as far as these may be necessary for the points 3, 6 and 10 in the present problem.

The points 4, 7 and 10 would have been somewhat better, but those chosen are quite suitable for good interpolations of intermediate values.

The preliminary investigation for the crown, the points 3, 6 and 10, and the springing, is given in Table No. 1, the results of which indicate fully to what extent the assumed design was in error.

In the preliminary diagram certain values of D and y_c were assumed. These are tabulated, together with the results from Table No. 1.

PLATE II.
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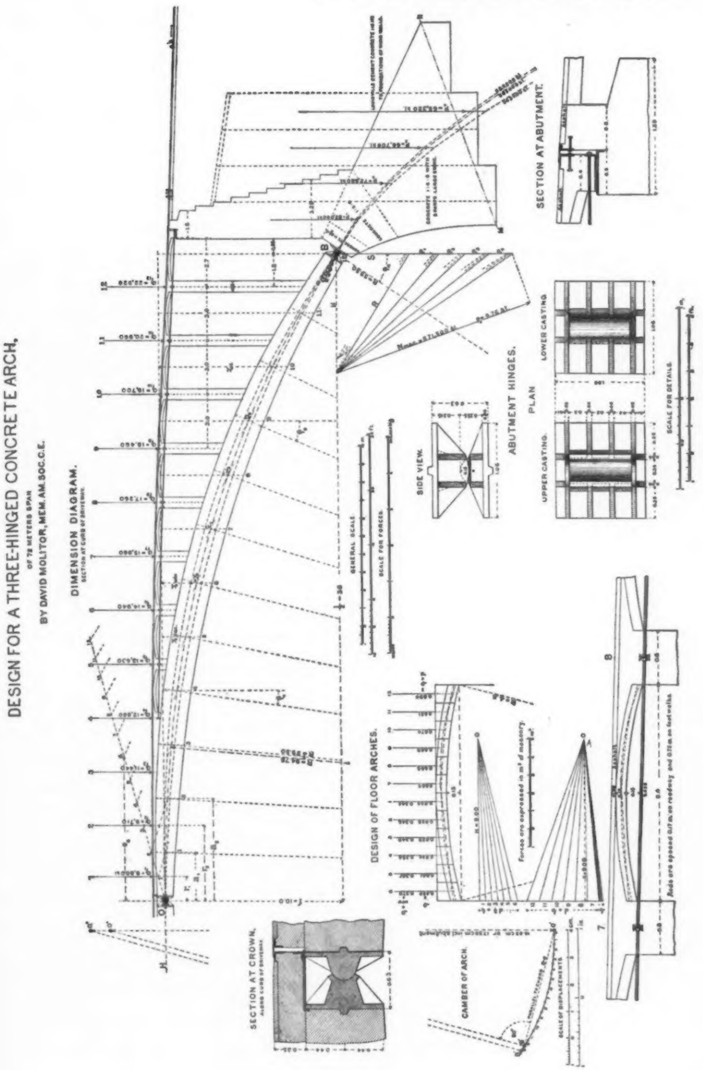
MOLITOR ON THREE-HINGED MASONRY ARCHES.



Point.	Preliminary Values.		Computed Values.	
	D	y_c	D	y_c
	m.	m.	m.	m.
0	0.82	0	0.880	0
3	1.40	0.38	1.436	0.423
6	1.63	1.90	1.660	1.964
10	1.40	6.30	1.500	6.394
12	1.00	10.00	1.060	10.000

From this comparison, which shows a remarkably close first approximation of shape and thickness of arch ring, it will be seen that it is now perfectly safe to proceed to the final design and the detailed computation thereof. This could have been done even with a less satisfactory coincidence in the above values of D and y_c.

From the values of D and y_c just found in Table No. 1, the intermediate values must now be interpolated, preparatory to constructing the final dimension diagram on Plate II.

To find the arch center line, plot the five points, whose co-ordinates a and y_c are now known, and connect them by circular arcs using the method indicated in Fig. 17. Connect the plotted points by straight lines forming the polygon $O - 3 - 6 - 10 - 12$. Then inscribe in any circle with convenient center $\times$ on the vertical, through O, a polygon $O' - 3' - 6' - 10' - 12'$, having its sides respectively parallel to those of the foregoing polygon. The radii $\times - 3'$, $\times - 6'$, etc., will be parallel to the required radii $\times_3 - 3$, $\times_6 - 6$, etc., and the latter, when drawn, will intersect in the centers $\times_3$, $\times_6$, $\times_{10}$ and $\times_{12}$ of the arcs sought. Usually a three-center curve will be found to fit an arch of this type, as was done in the present example (see Plate II).

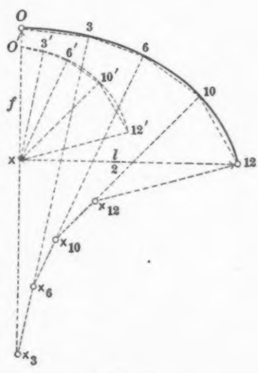


FIG. 17.

To interpolate values of D between those already found, plot the found values as ordinates with the corresponding values of a as in Fig. 18, choosing a vertical scale about twenty times the horizontal. By joining these points with the use of an irregular curved ruler, and noting that maximum D should occur at $0.3\,l$ from the crown, or at

about point 7, the intermediate values of D may then be scaled from the diagram with considerable accuracy.

The diagram, Plate II, can now be drawn, and all necessary dimensions for the final computation are then scaled therefrom. Any slight differences that may be found between these scaled dimensions and the final computed arch dimensions will be too small to warrant a reconsideration, especially as the knowable accuracy with which the dead loads can be determined is far less than any differences still to be expected at this stage of the solution. The final drawings should, of course, be constructed with the use of the dimensions resulting from the final computation.

The final computation is carried out in complete detail in Tables Nos. 2, 3, 4 and 5, and all the steps in the solution of the formulas used are readily traced without any further description. The weights q are based on an assumed weight of the concrete of 2 300 kgr. per cubic meter.

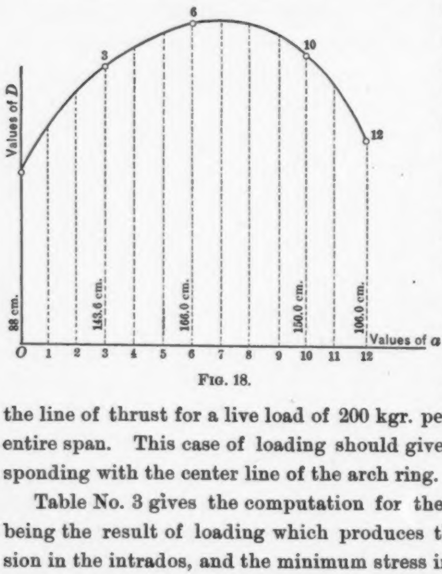


FIG. 18.

Table No. 2 gives the general data relative to dead loads, and the computation of the line of thrust for a live load of 200 kgr. per square meter over the entire span. This case of loading should give a line of thrust corresponding with the center line of the arch ring.

Table No. 3 gives the computation for the values of maximum y, being the result of loading which produces the maximum compression in the intrados, and the minimum stress in the extrados.

Table No. 4 gives the computation for the values of minimum y, being the result of loading which produces the maximum compression in the extrados and the minimum stress in the intrados.

Table No. 5 gives the computation of the thickness of the arch ring at the several points selected, also of the ordinates of the center line, and the unit stresses in the extreme fibers of the arch ring. See Plate II for graphical representation.

TABLE No. 1.—Preliminary Investigation. Live Load of 400 Kgr. per Meter, and Concentrated Load of 10 000 Kgr.

Point.	q	r	$q\,r$	Σq	$\Sigma q\,r$	$\frac{\Sigma q r}{\Sigma q} = b$	e	a	β	From Eqs. (32). H	V	$y_{max.}$	S Eq. (33).	N_i Eq. (36).	From Eqs. (34). H	V	$y_{min.}$	S Eq. (35).	N_e Eq. (36).	D Eq. (40).	w Eq. (41).	y_c Eq. (42).
	kgr.	m.	kgr. m.	kgr.	kgr. m.	m.	m.	m.		kgr.	kgr.	m.	kgr.	kgr.	kgr.	kgr.	m.	kgr.	kgr.	cm.	cm.	m.
1..	8 000	1.35	10 800	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
2..	9 120	4.2	38 304	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
3..	10 840	7.2	78 048	27 960	127 152	4.55	4.2	8.7	5° 40′	312 198	7 670	0.621	37 313	314 320	302 652	6 600	0.207	23 160	303 424	143.6	21.5	0.423
4..	12 400	10.2	126 480	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
5..	13 700	13.2	180 840	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
6..	14 600	16.2	236 520	68 660	670 992	9.77	8.3	17.7	15° 10′	314 637	6 993	2.266	78 495	324 112	295 713	4 673	1.621	67 747	303 045	166.0	33.0	1.964
7..	15 910	19.2	305 472	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
8..	17 220	22.2	382 284	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
9..	18 120	25.2	456 624	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
10..	19 080	28.2	538 056	138 990	2 353 428	16.93	13.0	29.7	26° 30′	317 020	6 330	6.590	150 526	350 550	287 330	2 344	6.150	143 326	320 796	150.0	22.0	6.394
11..	20 320	31.2	633 984	......	......	......	....	....	......	......	......	......	......	......	......	......	......	......	......	....	....	....
12..	22 840	34.2	781 128	182 150	3 768 540	20.689	....	36.0	33° 30′	......	......	......	......	......	......	......	......	......	......	....	....	....

$\Sigma_o^{\frac{l}{2}}\, q = 182\ 150.$ $s = 15.311.$

Constants............ $\frac{s}{f}\,\Sigma_o^{\frac{l}{2}}\, q = 278\ 890.$ $\frac{p\,l^2}{8} = 259\ 200.$ $W\left(\frac{l}{2} - 1\right) = 350\ 000.$

Case of maximum loading. Entire span covered with 400 kgr. per meter and 10 000 kgr. at the crown.

Equation (26)........ $H_{max} = \frac{s}{f}\,\Sigma_o^{\frac{l}{2}}\, q + \frac{p\,l^2}{8\,f} + \frac{W\,l}{4\,f} = 322\ 810$ kgr., and D_o at crown $= \frac{H_{max}}{k} = \frac{3228.1}{40}$ $= 80.7$ cm. $+ 10\% = 88$ cm.

Equation (27)........ $S_{max} = \Sigma_o^{\frac{l}{2}}\, q + \frac{p\,l}{2} + \frac{W}{2} = 201\ 550$ kgr.

Equation (36) .... $N_{max} = H_{max} \cos \beta_{12} + S_{max} \sin \beta_{12} = 394\ 760$ kgr., and D_{12} at springing $= \frac{N_{max}}{k} = \frac{3947.6}{40} = 98.7$ cm., or say 106 cm.

TABLE No. 2.—GENERAL DATA RELATIVE TO DEAD LOADS AND COMPUTATION OF LINE OF THRUST FOR LIVE LOAD OF 200 KGR. PER SQUARE METER OVER BRIDGE.

Point.	q	r	$q\ r$	$\Sigma_o^a\ q$	$\Sigma\ q\ r$	$\frac{\Sigma\ q\ r}{\Sigma\ q} = b$	a	e	β	Sin β	Cos β	$a - b$	$(a-b)\Sigma_o^a\ q$	$\frac{a^2\ p}{2}$	y Eqs. (26).
	kgr.	m.	kgr. m.	kgr.	kgr. m.	m.	m.	m.				m.	kgr. m.	kgr. m.	m.
1......	8 800	1.35	11 880	8 800	11 880	1.35	2.7	1.4	1° 00′	0.017	0.9998	1.35	11 880	729	0.042
2......	9 710	4.2	40 782	18 510	52 662	2.86	5.7	2.8	2° 50′	0.050	0.999	2.84	52 568	3 249	0.185
3......	11 440	7.2	82 368	29 950	135 030	4.51	8.7	4.2	5° 10′	0.090	0.996	4.19	125 490	7 569	0.442
4......	12 860	10.2	131 172	42 810	266 202	6.22	11.7	5.6	8° 10′	0.142	0.990	5.48	234 599	13 689	0.824
5......	13 630	13.2	179 916	56 440	446 118	7.90	14.7	6.9	11° 10′	0.194	0.981	6.80	383 792	21 609	1.346
6......	14 940	16.2	242 028	71 380	688 146	9.65	17.7	8.1	14° 00′	0.242	0.970	8.05	574 609	31 329	2.012
7......	15 980	19.2	306 816	87 360	994 962	11.39	20.7	9.3	16° 50′	0.290	0.957	9.31	813 322	42 849	2.842
8......	17 260	22.2	383 172	104 620	1 378 134	13.17	23.7	10.5	20° 10′	0.345	0.939	10.53	1 101 649	56 169	3.844
9......	18 460	25.2	465 192	123 080	1 843 326	14.98	26.7	11.6	23° 20′	0.396	0.918	11.72	1 442 498	71 289	5.026
10......	19 700	28.2	555 540	142 780	2 398 866	16.80	29.7	12.7	26° 30′	0.446	0.895	12.90	1 841 862	88 209	6.408
11......	20 960	31.2	653 952	163 740	3 052 818	18.64	32.7	13.8	29° 50′	0.497	0.867	14.06	2 302 184	106 929	7.998
12......	22 520	34.2	770 184	186 260	3 823 002	20.525	36.0	14.8	33° 30′	0.552	0.834	15.475	2 882 374	129 600	10.000

$$s = \frac{l}{2} - b_{12} = 15.475$$

Case of maximum loading, for live load $p = 400$ kgr. per square meter over entire span, and $W = 10\ 000$ kgr. at crown.

Constants $\quad \frac{s}{f}\ \Sigma_o^{\frac{l}{2}}\ q = 288\ 237 \qquad p\,\frac{l^2}{8} = 259\ 200 \qquad W\left(\frac{l}{2} - 1\right) = 350\ 000.$

From equation (26). $H_{max} = \frac{s}{f}\ \Sigma_o^{\frac{l}{2}}\ q + p\frac{l}{8} + \frac{W l}{4 f} = 332\ 157$ kgr., from which $D_o = \frac{3321.57}{40} = 83$ cm. 88 cm. was taken.

From equation (27). $S_{max} = \Sigma_o^{\frac{l}{2}}\ q + p\frac{l}{2} + \frac{W}{2} = 205\ 660$ kgr.

From equation (36). $N_{max} = H_{max} \cos \beta_{12} + S_{max} \sin \beta_{12} = 390\ 543$ kgr., from which $D_{12} = \frac{3905.4}{40} = 97.6$ cm. 106 cm. was taken.

From equation (37). $T \quad = S \cos \beta - H \sin \beta = -11\ 860$ kgr.

TABLE No. 3.—VALUES OF y_{max}, N_i AND T_i.

Point.	$\frac{pe}{2l}(l-e)$	$\frac{pe}{4f}(l-e)$	$\frac{p}{2l}\left(\frac{l}{2}+e\right)^2$	$pe\left(a-\frac{e}{2}\right)$	aV	$(a-b)\Sigma_o^a q$. from Table No 2.	H	$y_{max.}$	$\Sigma_o^a q$. from Table No. 2.	S	$H\cos\beta$	$S\sin\beta$	N_i	T_i
					kgr. m.	kgr. m.	kgr.	m.	kgr.	kgr.	kgr.	kgr.	kgr.	kgr.
1.....	274.0	986	3 884	2 160	22 105	11 880	319 683	0.113	8 800	17 545	319 621	298	319 919	12 106
2.....	538.3	1 938	4 181	4 816	45 159	52 568	320 635	0.320	18 510	27 552	320 315	1 377	321 692	11 493
3.....	791.0	2 847	4 488	11 088	66 729	125 490	321 544	0.632	29 950	39 299	320 258	3 537	323 785	10 203
4.....	1 032.9	3 718	4 806	19 936	86 909	234 599	322 415	1.059	42 810	52 477	319 191	7 452	326 643	6 169
5.....	1 247.8	4 492	5 111	30 050	106 034	383 792	323 189	1.609	56 440	66 412	317 048	12 884	329 932	2 452
6.....	1 437.8	5 176	5 401	44 226	124 311	574 609	323 873	2.294	71 380	81 642	314 157	19 757	333 914	816
7.....	1 619.8	5 831	5 699	59 706	141 613	813 322	324 528	3.127	87 360	97 920	310 573	28 397	338 970	— 404
8.....	1 793.8	6 458	6 004	73 290	158 013	1 101 649	325 155	4.099	104 620	115 485	305 320	39 842	345 162	— 3 738
9.....	1 946.3	7 006	6 292	96 976	173 942	1 442 498	325 703	5.261	123 080	134 233	298 995	53 156	352 151	— 5 752
10.....	2 092.0	7 531	6 586	118 618	189 159	1 841 862	326 228	6.590	142 780	154 227	291 974	68 785	360 759	— 7 465
11.....	2 231.0	8 032	6 887	142 416	203 721	2 302 184	326 729	8.106	163 740	175 488	283 274	87 217	370 491	—10 236
12.....	2 351.6	8 466	7 166	169 312	219 938	2 882 374	327 163	10.000	186 260	198 287	272 854	109 454	382 308	—15 223

EQUATIONS USED.

Equations.... (32)

$$\begin{cases} V = \frac{1}{l}\left[\frac{pl^2}{8} + W\left(\frac{l}{2}-1\right) - \frac{pe}{2}(l-e)\right] = 8\,461 - \frac{pe}{2l}(l-e) \\ H = \frac{s}{f}\Sigma_o^{\frac{l}{2}} q + \frac{1}{2f}\left[\frac{pl^2}{8} + W\left(\frac{l}{2}-1\right) + \frac{pe}{2}(l-e)\right] = 318\,697 + \frac{pe}{4f}(l-e) \\ y_{max} = \dfrac{(a-b)\,\Sigma_o^a q + aV + ep\left(a-\frac{e}{2}\right)}{H} \end{cases}$$

Equation..... (33) $S = \frac{p}{2l}\left(\frac{l}{2}+e\right)^2 + \frac{W}{l}\left(\frac{l}{2}-1\right) + \Sigma_o^a q.$

Equation..... (36) $N_i = H\cos\beta + S\sin\beta.$

Equation..... (37) $T_i = S\cos\beta - H\sin\beta.$

DATA.

Values of a, e, $\sin\beta$ and $\cos\beta$ from Table No. 2.

$\frac{l}{2} = 36$ m. and $f = 10.0$ m.

$p = 400$ kgr. per square meter and $W = 10\,000$ kgr.

$\frac{s}{f}\Sigma_o^{\frac{l}{2}} q = 288\,237$ and $\frac{W}{l}\left(\frac{l}{2}-1\right) = 4\,861.$

TABLE No. 4.—VALUES OF y_{min}, N_e AND T_e.

Point.	$\frac{p}{2}\left(\frac{l}{2}-e\right)^2$	$W\left(\frac{l}{2}-a\right)$	V	H	$p\,(a-e)$	$\frac{p}{2}\,(a-e)^2$	$(a-b)\,\Sigma_o^a q$ from Table No. 2.	aV	$y_{min.}$	$\Sigma_o^a q$ from Table No. 2.	S	$H \cos \beta$	$S \sin \beta$	N_e	T_e
			kgr.	kgr.			kgr. m.	kgr. m.	m.	kgr.	kgr.	kgr.	kgr.	kgr.	kgr.
1	239 432	333 000	7 950	316 859	520	338	11 880	21 465	—0.029	8 800	1 370	316 797	23	316 820	—4 016
2	220 448	303 000	7 270	314 409	1 160	1 682	52 568	41 439	+0.047	18 510	12 400	314 095	620	314 715	—3 332
3	202 248	273 000	6 600	311 999	1 800	3 872	125 490	57 420	0.231	29 950	25 150	310 751	2 263	313 014	—3 030
4	184 832	243 000	5 943	309 629	2 440	7 442	234 599	69 533	0.557	42 810	39 307	306 533	5 582	312 115	—5 053
5	169 362	213 000	5 310	307 355	3 120	12 168	383 792	78 057	1.034	56 440	54 250	301 515	10 524	312 039	—6 408
6	155 682	183 000	4 704	305 171	3 840	18 432	574 609	83 261	1.670	71 380	70 516	296 016	17 065	313 081	—5 450
7	142 578	153 000	4 105	303 016	4 560	25 992	813 322	84 973	2.489	87 360	87 815	289 986	25 466	315 452	—3 835
8	130 050	123 000	3 515	300 889	5 280	34 848	1 101 649	83 305	3.500	104 620	106 385	282 535	36 703	319 238	—3 911
9	119 072	93 000	2 945	298 840	6 040	45 602	1 442 498	78 631	4.713	123 080	126 175	274 335	49 965	324 300	—2 512
10	108 578	63 000	2 383	296 816	6 800	57 800	1 841 862	70 775	6.162	142 780	147 197	265 650	65 650	331 300	—1 219
11	98 568	33 000	1 827	294 815	7 560	71 442	2 302 184	59 743	7.848	163 740	169 473	255 605	84 228	339 833	+ 410
12	89 888	000	1 248	292 731	8 480	89 888	2 882 374	44 928	10.000	186 260	193 492	244 138	106 807	350 945	— 215

EQUATIONS USED.

Equations 34..

$$V = \frac{1}{l}\left[\frac{p}{2}\left(\frac{l}{2}-e\right)^2 + W\left(\frac{l}{2}-a\right)\right]$$

$$H = \frac{s}{f}\,\Sigma_o^{\frac{l}{2}} q + \frac{1}{2f}\left[\frac{p}{2}\left(\frac{l}{2}-e\right)^2 + W\left(\frac{l}{2}-a\right)\right]$$

$$y_{min} = \frac{(a-b)\,\Sigma_o^a\, q - a\,V + \frac{p}{2}\,(a-e)^2}{H}$$

Equation 35... $S = \Sigma_o^a q + p\,(a-e) - V.$

Equation 36... $N_e = H \cos\beta + S \sin\beta.$

Equation 37... $T_e = S \cos\beta - H \sin\beta.$

DATA.

Values of a, e, $\sin\beta$ and $\cos\beta$ are taken from Table No. 2.

$\frac{l}{2} = 36$ m. and $f = 10.0$ m.

$p = 400$ kgr. per square meter and $W =$ 10 000 kgr.

$\frac{s}{f}\,\Sigma_o^{\frac{l}{2}} q = 288\,237.$

TABLE No. 5.—VALUES OF D, y_c AND STRESS k IN THE ARCH RING.

Point.	Δy.	Cos β. Table No. 2.	c.	N_i Table No. 3.	N_e Table No. 4.	$\frac{N_e}{k\left(1+\frac{N_i}{N_e}\right)}$	$\left[\frac{N_i}{k\left(1+\frac{N_i}{N_e}\right)}\right]^2$	$6c\left[\frac{N_i}{k\left(1+\frac{N_i}{N_e}\right)}\right]$	D.	$\frac{kD}{N_e}-1$	w.	$y_{min.}$ Table No. 4.	y_c	w^1	UNIT STRESSES CAUSED BY N_e: $k_e=\frac{N_e}{D}\left(1+\frac{6w}{D}\right)$	UNIT STRESSES CAUSED BY N_e: $k_i=\frac{N_e}{D}\left(1-\frac{6w}{D}\right)$	UNIT STRESSES CAUSED BY N_i: $k_e=\frac{N_i}{D}\left(1-\frac{6w'}{D}\right)$	UNIT STRESSES CAUSED BY N_i: $k_i=\frac{N_i}{D}\left(1+\frac{6w'}{D}\right)$
	cm.		cm.	kgr.	kgr				cm.		cm.	m.	m.	cm.	atm.	atm.	atm.	atm.
1..	14.2	0.999	14.2	3 200	3 168	39.8	1 584.0	3 390.9	110.4	0.39	7.2	−0.029	0.043	7.0	+39.9	+17.50	+17.97	+40.0
2..	27.3	0.999	27.3	3 217	3 147	39.8	1 584.0	6 519.2	129.8	0.65	14.1	+0.047	0.188	13.2	40.0	+ 8.48	+ 9.67	40.0
3..	40.1	0.996	40.1	3 238	3 130	39.8	1 584.0	9 575.8	145.4	0.86	20.8	0.231	0.440	19.3	40.0	+ 3.00	+ 2.21	40.0
4..	50.2	0.990	50.1	3 266	3 121	39.9	1 592.0	11 998.9	156.4	1.00	26.1	0.557	0.820	24.0	40.0	− 0.06	+ 1.67	40.0
5..	57.5	0.981	56.4	3 299	3 120	40.1	1 608.1	13 569.8	163.3	1.10	29.9	1.084	1.339	26.5	40.1	− 1.89	+ 0.53	39.9
6..	62.4	0.970	60.5	3 339	3 131	40.4	1 632.2	14 665.2	168.0	1.15	32.2	1.670	2.002	28.3	40.0	− 2.80	− 0.20	40.0
7..	63.8	0.957	61.1	3 390	3 154	40.8	1 664.6	14 957.3	169.7	1.15	32.5	2.489	2.828	28.6	40.1	− 2.80	− 0.20	40.0
8..	59.9	0.939	56.2	3 452	3 192	41.4	1 714.0	13 960.1	166.6	1.09	30.3	3.500	3.833	25.9	40.0	− 1.72	+ 1.44	39.9
9..	54.8	0.918	50.3	3 522	3 243	42.2	1 780.8	12 735.9	162.7	1.01	27.3	4.713	5.010	23.0	40.0	− 0.14	+ 3.24	40.0
10..	42.8	0.895	38.3	3 608	3 313	43.2	1 866.2	9 927.4	151.8	0.83	21.0	6.162	6.397	17.3	39.9	+ 3.61	+ 7.60	40.0
11..	25.8	0.867	22.4	3 705	3 398	44.3	1 962.5	5 953.9	133.3	0.57	12.7	7.848	7.994	9.8	40.0	+10.97	+15.57	40.0
12..	00	0.834	00	3 823	3 509	......	......	......	......	......	0.0	10.000	10.000	0.0	......	......	......	......

EQUATIONS USED.

$$\triangle y = y_{max} - y_{min}, \text{ and } c = \triangle y \cos \beta.$$

Equation (40). $$D = \sqrt{6c\left[\frac{N_i}{k\left(1+\frac{N_i}{N_e}\right)}\right] + \left[\frac{N_i}{k\left(1+\frac{N_i}{N_e}\right)}\right]^2} + \frac{N_i}{k\left(1+\frac{N_i}{N_e}\right)}$$

Equation (41). $$w = \frac{D}{6}\left(\frac{kD}{N_e} - 1\right)$$

Equation (42). $$y_c = y_{min} + \frac{w}{\cos \beta}$$

Equation (39). $$k_1 = \frac{N}{D}\left(1 - \frac{6\,w}{D}\right) \text{ and } k_2 = \frac{N}{D}\left(1 + \frac{6\,w}{D}\right)$$

The values of N_i and N_e in the above table are given for an arch 1 cm. wide, thus giving unit stresses in atmospheres of 1 kgr. per cm.² for k_i and k_e.

w = distance from the point of application of N_e to the arch center line.

$w' = c - w$ = distance from the point of application N_i to the arch center line.

y_c = ordinate of the arch center line.

k = allowable unit stress on the extreme fiber = 40 atm. in compression; 2 atm. in tension (See also Fig. 16).

The pressure on the abutment foundation is given on Plate II.

TABLE No. 6.—CHANGE IN LENGTH OF THE ARCH RING.

Point.	$\Sigma_o^a q$ from Table No. 2.	$a\,p$	S	$H \cos \beta$	$S \sin \beta$	N	F	L	$\frac{N}{F}$	$\frac{NL}{F}$	$\frac{NL}{F}\left(\frac{1}{E}+\frac{1}{E'}\right)$	$L\,(\varepsilon + \alpha\,t)$.	$\Delta\ L$.
	kgr.	kgr. m.	kgr.	kgr.	kgr.	kgr.	cm².	cm.	atm.		cm.	cm.	cm.
0......	000	00	00	301 197	000	301 197	9 360	135	32.2	4 347	—0.0162	—0.1499	—0.1661
1......	8 800	540	9 340	301 137	159	301 296	11 040	285	27.3	7 780	—0.0290	—0.3163	—0.3453
2......	18 510	1 140	19 650	300 896	983	301 879	12 980	300	23.3	6 990	—0.0260	—0.3330	—0.3590
3......	29 950	1 740	31 690	299 993	2 852	302 845	14 580	301	20.8	6 261	—0.0233	—0.3341	—0.3574
4......	42 810	2 340	45 150	298 185	6 411	304 596	15 640	304	19.5	5 928	—0.0221	—0.3374	—0.3595
5......	56 440	2 940	59 380	295 474	11 520	306 994	16 330	306	18.8	5 753	—0.0214	—0.3397	—0.3611
6......	71 380	3 540	74 920	292 161	18 131	310 292	16 800	309	18.5	5 717	—0.0213	—0.3430	—0.3643
7......	87 360	4 140	91 500	288 246	26 535	314 781	16 970	314	18.5	5 809	—0.0216	—0.3485	—0.3701
8......	104 620	4 740	109 360	282 824	37 729	320 553	16 660	319	19.2	6 125	—0.0228	—0.3541	—0.3769
9......	123 080	5 340	128 420	276 499	50 854	327 353	16 270	325	20.1	6 532	—0.0243	—0.3607	—0.3850
10......	142 780	5 940	148 720	269 571	66 329	335 900	15 180	333	22.1	7 359	—0.0274	—0.3696	—0.3970
11......	163 740	6 540	170 280	261 138	84 629	345 767	13 330	342	26.9	9 200	—0.0342	—0.3796	—0.4138
12......	186 260	7 200	193 460	251 198	106 790	357 988	11 280	216	31.7	6 847	—0.0255	—0.2398	—0.2653

Average = 23.0

Total δ = —4.5208

EQUATIONS USED.

Equation..(26). $H = \frac{s}{f} \Sigma_o^{\frac{l}{2}} q + \frac{pl^2}{8f} = 301\ 197$ kgr. Equation (36). $N = H \cos \beta + S \sin \beta$.

Equation..(27). $S = \Sigma_o^a q + a p$. Equation..(46). $\triangle L = -\frac{NL}{F}\left(\frac{1}{E}+\frac{1}{E'}\right) - L\left(\varepsilon + \alpha\, t\right)$

A comparison of the last assumed data, and the final dimensions obtained in Table No. 5 is given in the following table, from which it is clearly seen that the last assumed arch ring gave dead loads and dimensions which could scarcely be improved by further computations:

Point.	Assumed Dimensions.		Computed Dimensions.		For ½ total live load as found in Table No. 2.
	D.	y_c	D.	y_c	y.
	cm.	m.	cm.	m.	m.
0	88	0.00	88	0.000	0.000
1	112	0.05	110.4	0.043	0.042
2	131	0.18	129.8	0.188	0.185
3	145	0.45	145.8	0.440	0.442
4	154	0.80	156.4	0.820	0.824
5	162	1.29	163.3	1.339	1.346
6	166	1.96	168.0	2.002	2.012
7	167	2.80	169.7	2.828	2.842
8	166	3.79	166.6	3.833	3.844
9	160	4.98	162.7	5.010	5.026
10	150	6.38	151.8	6.397	6.408
11	134	7.96	133.3	7.994	7.998
12	106	10.00	106.0	10.000	10.000

The above agreement between y_c and y illustrates very strikingly the statement previously made, viz.: that the line of thrust, for a case of entire span covered with one-half the uniform live load, represents practically the center line of the arch.

It will be noticed that the tensile stresses on the intrados at points 6 and 7 (see Table No. 5) are slightly in excess of the allowable, being 2.8 atm., as against 2 atm. allowed. This would probably be safe, but having set the limit at 2 atm., the excess of 0.8 atm. must be provided for, either by giving the section larger dimensions at these points, or by introducing a few iron rods to take up the excessive tension. The latter method will be adopted merely to illustrate the application of the formula.

The sectional area of iron required for an arch of 1 cm. width is by

$$\text{equation (45), } a = \frac{k_1^2 D \left(\frac{D}{2} - \frac{k_1 D}{3 (k_1 + k_2)}\right)}{2 f v (k_1 + k_2)}$$

in which $k_1 = 0.8$ atm., $k_2 = 40$ atm., $D = 169.7$ cm., $f = 700$ atm. (10 000 lbs. per square inch), and $v = 75$ cm., leaving about 10 cm. of concrete outside the iron.

By substituting these values in (45), and solving, it is found that $a = 0.002$ cm.^{2}, or for an arch 1 m. wide 0.2 cm.^{2} would be required,

or 3 rods of 3 mm. diameter for each meter of arch, or 7 rods of 2 mm. diameter, whichever may be preferable.

(*f*) *Design of the Hinged Bearings.*—The radius of curvature of the rolling surface according to Winkler, Heinzerling or Melan is given by $r = \frac{2\,N}{\pi\,k\,l}$ in which $N =$ the maximum normal pressure on the bearing $= 390\,543$ kgr. for 1 m. width of arch, $l =$ length of rolling surface $= 70$ cm. and $k =$ allowable unit working stress $= 240$ atm. for cast iron under slow motion.

Then

$$r = \frac{2 \times 390\,543}{3.14 \times 240 \times 70} = 15 \text{ cm.}$$

To find the height of the bearing necessary to distribute the stress over the concrete, let

$h =$ height of the bearing.

$D =$ thickness of arch ring adjacent to the bearing $= 106$ cm.

$b =$ width of bearing $= 100$ cm., but the rolling surface is only 70 cm.

$N =$ normal pressure on the bearing for 1 m. width of arch $= 390\,543$ kgr.

$T =$ tangential stress on the bearing for 1 m. width of arch $= 11\,860$ kgr.

$M =$ maximum bending moment on the bearing.

$I = \frac{b\,h^3}{12} =$ moment of inertia of vertical section through axis of roller.

$k =$ allowable unit stress $= 500$ atm. for cast iron, quiescent load.

Then

$$M = \frac{N}{D} \times \frac{D^2}{8} = \frac{N\,D}{8}; \text{ also } M = \frac{k\,I}{\frac{h}{2}} = \frac{N\,D}{8}$$

which solved for h gives

$$h = \sqrt{\frac{3\,N\,D}{4\,k\,b}} = \sqrt{\frac{3 \times 390\,543 \times 106}{4 \times 500 \times 70}} = 30 \text{ cm.}$$

The coefficient of friction between cast iron and stone is about 0.6; hence a tangential stress of $0.6\,N = 234\,300$ kls. would be required to

slide the bearing plate. The existing tangential stress is only 11 860 kgr. However, a small rib is provided on the bottom of the plate to prevent sliding during the erection of the structure.

To diminish the sliding friction between the roller surfaces of the bearings, it is intended to make the radius of the convex surface 15 cm. and that of the concave surface 16.5 cm. (not shown on the drawings), thus converting the sliding friction into rolling friction. The arc described by these rollers, for the extreme movements of the arch, is so small that the point of contact between the rolling surfaces would not be appreciably displaced, and there is absolutely no danger of unequal distribution of pressure on the casting, even were the motion to reach 1° of arc.

(*g*) *Composition of the Concrete.*—The full section of the arch ring for a distance of 2 m. from the hinges, and all outside the middle third of the arch ring; also the floor arches and the exposed surfaces of abutments and piers for a depth of 20 cm. from the surface, shall be of concrete, composed of 1 part Portland cement to 2 parts sand to 3 parts limestone. The middle third of the arch, the surfacing of the roadway under the asphalt composition and the cores of the small piers shall be of concrete, composed of 1 to 3 to 6. All other concrete shall be mixed 1 to 4 to 8 except the abutment foundations which shall be made as shown on Plate II.

(*h*) *Camber.*—The camber to be allowed in the arch ring will now be found for the condition that the bridge when completed, carrying the dead load and a live load of 200 kgr. per square meter, shall be at its true level at 0° Cent.

To realize this, the falsework must be superelevated by an amount equal to this camber plus the settlement which the former may undergo up to the time of closing the arch ring. The settlement in the falsework should be determined by actual tests made prior to construction.

After the arch ring is closed at mean temperature of, say, 24° Cent., and under no stress, it should be above its geometrical shape by the amount of the camber. Hence, the camber will be equal to the deflection at the crown caused by the dead load and the uniform live load of 200 kgr. per square meter, and a diminution in temperature of 24° Cent. below the mean. Then, under ordinary temperatures and loads, the arch will usually be above its theoretical position, which is very

desirable, as the horizontal thrust is materially increased by a diminution in the rise of the arch.

The thrusts N resulting from the assumed case of loading are found in Table No. 6, using equations (26), (27) and (36). The shortening in the successive arch sections between the crown and the abutment, for the respective values of N, are also found in Table No. 6 from equation (46). As the assumed case of loading is one of symmetry, only the half arch is treated.

The concrete to be used for the arch ring should possess an ultimate compressive strength of 220 atm. at probably the age when the bridge will be first tested. From Table No. 6 the average unit working stress, for the case of loading just above mentioned, will be seen to be 23 atm. Then the values of ε, α, E and E' can be taken from the data given under Section $V(d)$, as follows: $\varepsilon = 0.0009$, $\alpha = 0.0000088$, $E = 295\,000$ atm., and $E' = 3\,000\,000$ atm. For $t = -24^\circ$ Cent., $\varepsilon + \alpha\, t = 0.00111$ and $\frac{1}{E} + \frac{1}{E'} = 0.00000372$, which values are used in Table No. 6 to find the $\triangle\, L$'s.

The abutments themselves will be somewhat displaced as a result of stress and temperature effect, though the shrinkage will probably have taken place prior to closing the arch ring. Hence the equation for the shortening in the abutment may be written $\triangle\, L = -\frac{N\,L}{F}\left(\frac{1}{E} + \frac{1}{E^1}\right) - \alpha\, t\, L$. Taking $F = 69\,000$ cm.^{2} as an average value, and $L = 1\,000$ cm., $\triangle\, L$ becomes 0.23 cm. This displacement is considered in the graphical solution on Plate II, but not in the analytical solution here following:

From Table No. 6, the value of $\delta = \Sigma_0^{\frac{l}{2}} \triangle\, L = 4.521$ cm., of which 3.693 cm. is permanent and 0.828 cm. is elastic.

Also $n = \Sigma_0^{\frac{l}{2}} L = 3\,789$ cm., $f = 1\,000$ cm., and $\frac{l}{2} = 3\,600$ cm.

Hence from equation (49),

$$C_1 = \left(1 - \frac{\delta}{n}\right)\sqrt{f^2 + \frac{l^2}{4}} = 3\,731.85 \text{ cm.}$$

and from equation (50),

$$\triangle f = f - \sqrt{C_1^2 - \frac{l^2}{4}} = 16.79 \text{ cm.}$$

This agrees very closely with the value 16.65 cm., obtained from the graphical solution on Plate II.

When the displacement of the abutments, amounting to 0.23 cm., is included, the total deflection at the crown will be 17.5 cm. The deflections at any other points of the arch ring may be scaled from the diagram, Plate II.

If the falsework were absolutely rigid, the crown of the arch would require a superelevation of 17.5 cm., so that the arch, if closed at 24° Cent., and when carrying its own weight and a live load of 200 kgr. per square meter, will have a rise of 10 m. at 0° Cent. The design of the falsework is not made a part of this problem.

(*i*) *The Pressure on the Abutment Foundations.*—This is found to be 4.76 atm. (see Plate II). This, for the character of the substrata, assumed in the problem as hard clay, is in no way excessive. However, when dealing with a specific case, the pressure area may be made any desired quantity and the foundation pressure be diminished to such intensity as may seem safe for the particular case.

The fact that small settlements in the abutments of three-hinged masonry arches are not attended by any serious consequences, especially when sufficient camber was put into the arch during construction, makes it perfectly safe to exceed the pressure limits, on foundations hitherto allowed for arches without hinges, by from 50 to 100 per cent.

(*j*) *Estimate of Quantities.*—The following table contains the quantities of Portland cement concrete of the various mixtures for the different parts of the structure.

Structural parts.	Quantities of Concrete, in Cubic Meters.				
	1:2:3	1:2.5:4	1:3:6	1:4:8	1:4:8:3st.
Arch ring	978	....	478	....	....
Floor piers	165	....	52	....	....
Floor arches and floor	328	....	....	....	....
Two abutments and wing walls	141	....	....	1 661	....
Two abutment foundations	....	60	210	....	1 664
Totals in cubic meters	1 612	60	749	1 661	1 664
Totals in cubic yards	2 108.5	78.5	979.7	2 172.6	2 176.5

ESTIMATED QUANTITIES.

Items.	Quantities.	
	Metric units	U. S. units.
Portland cement concrete, mixed, 1c. : 2 s. : 3 broken stone...	1 612 m^3.	2108.5 cu. yds.
" " " " 1 : 2.5 : 4 " " ...	60 "	78.5 "
" " " " 1 : 3 : 6 " " ...	749 "	979.7 "
" " " " 1 : 4 : 8 " " ...	1 661 "	2172.6 "
" " " " 1 : 4 : 8 b. s., 3 stone....	1 664 "	2176.5 "
Louisville cement concrete 1 : 6 : 12 broken stone..	720 "	941.8 "
Earth excavation..........	5 400 "	7063.0 "
Asphalt pavement over abutments, 15 cm. concrete found...	176 m^2.	210.3 sq. yds.
Asphalt composition 6 cm. thick over bridge..........	600 "	717.0 "
Concrete footwalks over abutments..........	88 "	105.2 "
Concrete balustrade over abutments..........	44 m.	144.3 ft.
Iron hand railing over bridge..........	150 "	492.0 "
194 m. of steel grooved rail, 80 lbs. per yd. or 39.76 kgr. per m.	7 713 kgr.	16 970 lbs.
972 steel rods, 2.5 cm. dia. 3.05 m. long, nuts each end at 12.4 kgr	12 052 "	26 514 "
120 m. *I* at 14.9 kgr. per m. or 10 lbs. per ft..........	1 788 "	3 934 "
Metal in expansion joints of roadway..........	5 000 "	11 000 "
Cast-iron hinged bearings..........	81 600 "	179 520 "
Falsework..........	72 lin. m.	236.2 lin. ft.

Unit prices have not been inserted in the above table because these are too much dependent on local conditions and market values. However, a liberal estimate of cost for the entire structure as designed, including 10% for engineering contingencies, would be about $96 000.

DISCUSSION.

WALTER G. BERG, M. Am. Soc. C. E.—This subject deserves attention and study on account of its admirably clear presentation by the author, and, further, because several bridges have been built after this method in Germany by engineers who stand very high in the profession. However, it should be remembered, that this is a structure of a class prevalent in European practice, in the design of which the tendency of the engineer has been to consider a structure as perfect, in an engineering sense, if the sections have been reduced to the actual theoretical minimum, independent of the question of additional cost of labor, and other features which come in. In American practice the aim is simplicity, even at a sacrifice of additional material. The speaker coincides with Mr. Goldmark in his statement, that, by the addition of a few inches to the thickness of the ring, which in itself is a very small percentage of the entire masonry in the structure, absolute safety can be obtained without going into these theoretical detailed methods of construction. Perhaps the solution of the whole question would be to take extra care in making the foundations absolutely safe, and so do away with the main argument in favor of the three-hinged masonry arch, namely, that it should be used where the foundation might settle. The speaker does not wish to take the stand that nothing can be learned from European practice; but he believes that the tendency in America will always be toward a simple structure, even with the use of more material, thus avoiding the very great refinement required in a construction of this kind. Mr. Berg.

J. B. JOHNSON, M. Am. Soc. C. E.—Engineers in America have made and are likely to continue to make some mistakes in the matter of avoiding those structures, the computations for which are difficult. Careful distinction should be made between a difficult computation and an uncertain computation. The speaker believes that the ordinary arch is, under all circumstances, an uncertain product, because of the unknown rigidity of the abutments and the amount of deformation which occurs in the arch when the centering is removed. When a crack is seen, it is known that the tension at that place has been more than the masonry would stand. How far that crack extends is not known, and the pressures are not known. Mr. Johnson.

Now, the three-hinged masonry arch is a structure as stable as any, and the speaker, having as much confidence in Portland cement as in the best kind of building stone, believes that, humanly speaking, it may be regarded as absolutely permanent. These arches may be built either of stone or of concrete, with or without steel embodied in the arch ring, and they may be adapted to almost any set of conditions.

The author has given a very clear, concise, and accurate solution of the problem. The argument that the solution of each problem is difficult, tedious, or unknown to most engineers, has no value. Sup-

Mr. Johnson. pose the problem is of necessity long and tedious to the greatest expert, to the man in whom is placed the highest confidence; suppose he spends a month in figuring the stresses in such a structure—this would be said to be a tedious computation; but what is a month's expense of one man computing an important structure, as compared with the cost of the structure? It is simply nothing at all. The important questions are: Are his figures representative of the facts after the structure has been erected? And is it a structure which is readily erected in accordance with theory? The speaker believes that the three-hinged masonry arch can be readily erected in accordance with theory, and that the theory is competent to determine the facts. If this is so, such structures should be accepted as normal, as proper, and as having many advantages which the present iron and steel structures do not possess, one of which, the advantage of permanence, no one would claim for iron and steel in the present state of ignorance as to the proper methods of preserving them from rust.

Mr. Goldmark. HENRY GOLDMARK, M. Am. Soc. C. E.—This paper is an able argument in favor of a masonry arch—more distinctly a concrete arch—with three hinges. The argument is the stronger because it is enforced by a detailed design of a bridge built according to the formulas developed in the first part of the paper. The formulas in question look much more formidable than they really are. The arch being built with three hinges, the author assumes that the reactions at the crown and the springing will pass through the hinges, and that the two halves of the arch will turn freely about them. The problem of determining the stresses in any section becomes, under these conditions, a purely statical one, without the necessity of introducing the more complex methods involving the theory of elasticity. The solution of the stresses can readily be reached by simple analytical or graphical methods which, for any particular case, present no difficulty. The apparent complexity of the author's formulas is due to the fact that he presents the problem in a very elegant general form, in order to make the solution more complete for any possible case that may arise in practice. The discussion of the best way of arriving at the deflections the speaker has found of much value.

The interest of the paper centers more particularly on the question whether a masonry arch with hinges is so much better than one without hinges as to warrant the substitution of these details for an arch of ordinary form. As to this, opinions may very well differ. Masonry arches have been used, almost from time immemorial, and the results, as the author states, have been very favorable. The burden of proof in favor of introducing a new form rests, of course, entirely with those who advocate the innovation.

The author's reasons for wishing to introduce the hinge are, first, his belief that in case the foundations settle the hinged arch would be safer, and could be used where the foundations are not considered sufficiently uniform and solid for an ordinary arch; second, that the

stresses being more determinate, there would be less chance of the arch cracking; and third, that temperature changes would be of less importance in the jointed arch. Mr. Goldmark.

To the speaker it seems that the substitution of this form needs stronger evidence than is brought forward by this paper. The introduction of a movable joint appears, in his opinion, to be radically at variance with the character of masonry construction. As a matter of fact, even in the author's careful design, the idea is not fully carried out. The flooring and spandrel filling are really continuous and would oppose great resistance to motion at the hinge in the crown. The result would be that cracks would take place in the spandrel or else that the hinge would remain fixed. Either alternative would be unfortunate. In the latter case the author's assumption as to the determinate character of the stresses would be vitiated. It is true that the hinges would confine the line of pressure within somewhat closer limits, but the arch would, after all, be an elastic and not a static arch. Ordinary prudence would, therefore, prescribe that the usual practice be followed and the stresses determined, both for the fixed and the hinged end conditions, so that the dimensions of the arch ring could be proportioned for the more unfavorable case.

The design which the author works out with great care differs in no essential respect from similar arches which have been built without hinges, nor is the section of the arch which he finally obtains very different from that of ordinary arches which have stood well in practice.

The hinged arch, if the hinge acts, will, however, have greater deflection and hence involve greater stresses, at least in some parts of the structure. It is hardly conceivable, at least in the case of a railroad bridge, that this can be anything but detrimental. It may be added that the introduction of metallic parts, which are subject to corrosion, must be considered objectionable in a masonry bridge.

The experiments of the Austrian Society of Engineers and Architects, made some five or six years ago, to which the author refers, show clearly that the ordinary hingeless arch, whether built of concrete, brick or rubble, does act very much as the elastic theory indicates. This theory is complex in its application, almost too much so for every-day practical use, except in rare cases. At the same time there need be no great difficulty in determining perfectly safe dimensions for even the largest arches. Arch estimates, such as the speaker has lately had occasion to make on several large arches, show that the total amount of masonry in the arch-ring is, after all, only a small percentage of the total masonry in the bridge. The addition of 6 ins., or 1 ft., to the arch, above the absolute minimum required, would make the bridge safe beyond peradventure, without increasing its cost appreciably. The economy that could be introduced by the use of hinges is, then, not very great, while it would be more than counterbalanced by the accompanying drawbacks.

CORRESPONDENCE.

Mr. Lindenthal. GUSTAV LINDENTHAL, M. Am. Soc. C. E.—The writer agrees with the author that masonry arch bridges should be built in preference to iron and steel bridges wherever possible, because they will last longer and cost next to nothing for maintenance.

To argue from this, however, that masonry arches of long span are only feasible or preferable when built with hinges, is a conclusion with which the writer does not agree. Masonry arches have the very valuable qualities of rigidity and durability. These qualities are impaired by hinges. Unyielding foundations, rigid abutments and careful work should not be less necessary for hinged arches than for fixed arches, which are better in all respects. For none of the hinged masonry arches already built can it truthfully be claimed that they are as rigid, or as good, or as durable, as fixed masonry arches of corresponding material. If the Cabin John Arch were hinged, it would not be the absolutely rigid bridge it is.

Hinges may give a theoretical excuse for paring down the dimensions of the arch ring and abutments, and for building on "any moderately good foundation"; but such bridges will not stand the test of time with good proportioned and well-built fixed masonry arches. If the foundations are not secure, and cannot be made secure, no arch bridge of any kind should be attempted.

With the same proportion of rise (10 m.) to span (72 m.) as in the author's design (page 58), and for the same moving loads and same unit pressure, equal to $\frac{1}{12}$ of the crushing load on the arch ring, a span of 140 m. (459 ft.) is readily feasible. If the proportion of rise to span is $\frac{1}{4}$, a span of 600 ft. becomes feasible under the same conditions, but in all cases the writer would prefer a fixed arch to one with hinges.

The supposed greater accuracy of calculation of hinged arches, as compared with the ordinary fixed masonry arch, is largely illusory. The fact which is not illusory, however, is that the hinged bridge is less rigid than the fixed arch bridge. The question is not that it is rigid enough, or that it is more rigid than a metal bridge. Of rigidity in a bridge there cannot be too much; if the fixed arch bridge is more rigid than the hinged arch bridge, which it certainly is, that should be sufficient reason for avoiding hinges.

But there are other, if minor, considerations: The hinges, having to be of metal, require to be looked after and kept in good condition by cleaning and painting, thus introducing the trouble and cost of maintenance, entirely absent in the fixed stone arch bridge. Corrosion of the metal hinges is not preventable, and the durability of the bridge is limited by that of the hinges.

Mr. Lindenthal.

The refinement of having the concave surface of the hinge of somewhat larger radius than the convex surface, for the purpose of minimizing the friction, still more increases the liability to corrosion. Dust and rust will lodge in the hinge, where any amount of cleaning cannot remove them. Changes in the form of the arch, from whatever cause, will be localized at the hinges. Bending strains in the arch-ring are then unavoidable, and it is not possible to determine them with any definiteness. In the fixed arch, such bending strains are distributed, and, therefore, minimized, over its entire length.

The calculation of fixed masonry arch bridges presents no greater difficulties than that of hinged arch bridges.

The theory for the calculation of fixed masonry arches, based on the elasticity of the material, which the author (page 33), claims as the only one entitled to confidence, has not received general acceptance, and does not deserve it. In so far as any elasticity at all is ascertainable in building stone and mortar, it is so uneven, and between such very wide limits, that any theoretical results based on it must necessarily be equally unreliable and divergent. Elasticity has no bearing on the determination of the camber to meet the flattening of the arch after completion, which is an old arithmetical operation with data from experience on the compressibility of different kinds of masonry and foundations.

The stability and strength of stone arches of any form can reliably be determined upon the same principle which was used by the old masters who built the great German cathedrals of the middle ages. The finely dimensioned and exquisitely balanced arches and vaults, meeting on top of slender columns, were not the product of chance; they were determined by calculation, in which the "Bauwage" (a literal translation of which is building balance) was the principal instrument. It was simply a string or cord representing the center line of the arch. The string was loaded and shaped with weights representing the loads to be carried by the arch. One end of the string passed over a pulley and was balanced by a weight, which represented the horizontal thrust. The inverted equilibrium polygon, of course, corresponded in all respects to an erect one. Thus the position, intensity and direction of the thrust were known at every point. This principle, also used by Rankine as a basis for his theory of masonry arches, reaches back to antiquity, it being claimed that the cathedral builders derived it from Arabian mathematicians. It can be used with perfect safety, aided by the greater accuracy of modern analysis and graphical statics, for the largest masonry arch spans of any form.

The arch, assumed as carrying all the loads, vertically, of course, and not radially, is by this method treated as a bent wall, held in position and equilibrium between unyielding abutments. For such an arch, considered as an inert mass, the pressure line, as to form and

Mr. Lindenthal. location, can be ascertained with an accuracy sufficient for all practical purposes; as also can the variations in the pressure line from moving load. Rankine's rule, which is an excellent one, and should be adhered to in all cases, is that the pressure line and variations of same shall not pass out of the middle third of the arch-ring section, which should be thoroughly bonded. In flat arches this condition can always be complied with. For long spans the arch material may be arranged in the form of deep ribs.

Rigid spandrels, which can very well consist of open gallery work in longer spans, good cement and good bonding, in an arch, give additional security. Piers or walls are likewise, as a rule, so proportioned that the pressure line does not pass out of the middle third. The safety of all such masonry is assured, even if the coursed stones were laid without mortar, the latter merely insuring a good fit. Hinges for fixing the pressure line through the center line of the arch are not necessary; but rigid abutments, painstaking care and good judgment in the execution of the work are of greater importance, whether there are hinges or not.

As a rule, it is much easier to design and calculate a masonry arch than to execute it as a perfect work.

The great amount of bad work in arch building and in the foundations is as responsible for cracks and failures as bad designing. The effort and skill of the engineer should be directed toward getting not merely "moderately good foundations," which the author claims are good enough for hinged arches, but to get unyielding foundations. In that respect the better knowledge of physical conditions, the better materials, and the greater resources of modern engineering, can be used with greater advantage than was ever possible before; where these cannot be had, metal truss bridges are preferable.

Brick arches, made up of separate rings, each a single brick in thickness (instead of the arch ring being bonded through like a wall), with from four to eight such rings above, and unconnected with, each other, may be seen carrying railroad tracks. The rigidity of such an arch is derived only from the spandrel filling. Yet because such arches sometimes hold up (the bricks, as it were, having more sense than the builder that put them there), that cheap practice is frequently pointed out as a good example. It is bad work of this and similar kind that has discredited masonry arches.

For long span masonry arches, but always of the fixed kind, the writer would prefer granite of the best quality, rather than the best concrete, the stones to reach through the entire thickness of the arch ring in every third course at least. The flattening of the arch, when released from the staging, is thereby distributed over all the many radial joints. But if monolithic arch masonry of the quality proposed by the author is used, greater safety against cracking can

be had, in the judgment of the writer, by embedding laterally connected iron ribs, corresponding to the Melan construction. This system lends itself very well to deep ribs in long spans. That hinges cannot be used for skew arches is too obvious for argument. Mr. Lindenthal.

The writer disagrees with the author also in the advocacy of pressures as low as $\frac{1}{8}$ of the crushing strength of the arch material. Whether concrete, or granite, or other stone of high quality, their homogeneity, with all possible care, can never be so reliably secured as to risk a factor of safety equal to that in steel or wrought iron. One-sixth of the breaking strength would be low, even for cast iron, but for stone masonry of the best kind there is no justification for less than $\frac{1}{10}$, and it should be nearer to $\frac{1}{20}$.

The durability of stone bridges is to be ascribed to their high factor of safety, necessary as a margin against weathering, and against accidental loading in the form of shock in a material, which, for greater safety, should always resist only by compression and never by tension, and principally to the great proportion of mass to moving load.

Reduce this mass in the arches and in the foundations one-half or two-thirds, as advocated by the author, under the pretense of more accurate calculation where the coarseness and variation of the material naturally do not admit of it, and stone bridges will cease to be the synonyms of strength and durability. Hinges in masonry arches are not only an incongruity, but may without prejudice be considered as one of the fads from which even an exact science like engineering is not free, or, rather, as an application of science to a wholly useless, if not injurious, purpose.

David A. Molitor, M. Am. Soc. C. E.—The opinion frequently advanced, that the design of structures of the kind under consideration requires great refinement, and also that their theoretical treatment is more difficult and cumbersome than the methods commonly used in the analysis of fixed arches, is entirely unfounded. In fact, if a fixed arch were to receive a detailed analysis, such as is given in the the paper on hinged arches, this might be justly called complicated. Yet any construction, the design of which has not received such analysis, certainly does not deserve professional sanction. It is far more important to solve a problem with exactness, even at a somewhat greater expenditure of labor, than to risk failures of construction. Mr. Molitor.

The time is fast approaching when the wasteful American methods, variously referred to, will receive the severe condemnation to which they are generally entitled, and the "rule of thumb" engineer will probably share the fate of retirement.

A careful consideration of the paper will undoubtedly convince the reader that the method is not nearly so complicated as might at first

Mr. Molitor. appear, and that the analysis of a fixed arch would be far more tedious if considered in a more general manner with live loads as they actually occur. The commonly applied graphical solution is simple merely because the difficult complications are not dealt with, but supplanted by vague and often erroneous assumptions.

Another fact which has not been considered by the advocates of superfluous material, intended to cover the factor of ignorance, is that an arch of excessive thickness, and consequent great rigidity, is far more liable to crack as a result of shrinkage of the material and settling when centers are released than is a well-designed arch, with just sufficient material to carry the loads. Every arch is subject to large distortions during construction, and, unless provision is made therefor, cracks will show soon after completion.

Among the many large fixed masonry arches which the author has seen, both in Europe and America, not excluding those of his own design and construction, he can recall only a few which, in spite of the best foundations, did not show cracks. This does not mean that such arches are unsafe or that they will not be lasting, but such defects always reflect injuriously on the reputation of tho designer and builder, which is unpleasant, to say the least.

In no instance has the author asserted that it is impossible to construct long-span masonry arches without hinges, but he has advocated hinges for them simply because they are more susceptible to cracks, etc., than short-span arches.

Many are the schemes which have been advanced to overcome the injurious effects of settlement during construction, but most, if not all, of them are open to severe criticism. The introduction of hinged connections has offered the best solution of the difficulty.

According to some of the criticism offered, it would be improper to construct a masonry arch on any but bed-rock foundations, which would naturally make this style of structure one of rare occurrence. To build an arch on any moderately good foundation requires the highest engineering skill and experience, while any one familiar with the use of mason's tools might safely risk the construction of an arch on rock foundations.

The question of rigidity and deflections has been strongly emphasized, but this objection to hinged masonry arches is entirely erroneous, since in them the proportion of live to dead loads is so small that the introduction of hinges does not materially influence the rigidity, as far as the purpose of the structure is concerned. The argument applies only to metal arches where the imposed loads are excessive.

In advocating low unit working stresses for masonry and concrete, the author naturally referred to a structure, the stresses in which are fully determinable. To conclude therefrom that this would be per-

missible for a fixed arch where the stresses may readily exceed the computed values by 100%, or more, is perfectly ridiculous. Mr. Molitor.

Skew arches may be built according to the method outlined in the paper, provided the skew is very slight, otherwise internal stresses arising from the skew would be likely to exert a harmful influence.

The preservation of iron hinges is not a very serious objection to the use of this style of structure; since the entire hinge may be embedded in asphaltum or some hard fatty mixture which will in no wise interfere with the freedom of action of the hinge within necessary limits, but at the same time will prevent corrosion of the metal.

The impression seems to prevail that iron is the only material suitable for such hinges. This is not true. On the contrary, such hinges can be made of a hard stone, like granite, or lead joints may be used to advantage. The choice of the kind of material rests entirely with the designer, and in the example submitted the author used cast iron.

While thanking Mr. Goldmark for his complimentary remarks, it is to be regretted that he did not give the paper sufficient study to familiarize himself with the design, instead of accusing the author of having designed a hinged arch with fixed spandrel filling. Mr. Goldmark also states that the design submitted "differs in no essential respect from similar arches which have been built without hinges," etc. No argument, as to the essential differences will carry more conviction than a careful inspection of the drawings.

The points brought out by Mr. Lindenthal have, for the most part, received consideration in the above, and it is not deemed necessary to draw out this discussion for the purpose of refuting what appears to be a recitation of text book information and personal views and preferences not necessarily accepted by others.

The examples of hinged arches, cited in the first part of the paper, certainly outweigh most of the arguments and doubts expressed by those who have taken up this discussion.

It is not in making designs and estimates or in the study of text books that the great advantages of three-hinged masonry arches may be learned and appreciated, but the constructor who has sought in vain to produce a perfect masonry bridge without a defect or blemish, will know without doubt or much argument which style of structure is best suited to practical application.

In closing, the author expresses his warm appreciation of the able support received from so high an authority as Professor J. B. Johnson.